Chapter 2

Fundamental Mathematical Formulation for the Theory, Dynamics and Applications of Magnetic Resonance Imaging

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Fundamental Mathematical Formulation for the Theory, Dynamics and Applications of Magnetic Resonance Imaging

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Abstract

All Magnetic Resonance Imaging (MRI) techniques are based on the Bloch NMR flow equations. Over the years, researchers have explored the Bloch NMR equations to significantly improve healthcare for accurate diagnosis, prognosis and treatment of deceases. However, MRI scan is still one of the most expensive anywhere. Method to achieve the best image quality with the lowest cost is still a big challenge. In this chapter, the generalized time dependent non homogenous second order differential equation derived from the Bloch NMR flow equations is modeled into basic and well known equations such as Bessel *equation, Diffusion equation, Wave equation, Schrödinger's equation, Legendre's equation, Euler's equation and Boubaker polynomials. Solutions to these equations are abundantly available in standard text books and several research studies on Mathematics, Physics, Chemistry and Engineering. Unexpected NMR/MRI methodological developments may be possible based on the analytical solutions of these equations and may further enhance the power of NMR. There will be spectacular applications in a variety of fields, ranging from cognitive neuroscience, biomedical engineering, imaging-science, molecular imaging to medicine, and providing unprecedented insights into chemical, biological and geophysical processes. This may initiate unforeseen technological and biomedical possibilities based on a much improved understanding of nature.*

Keywords

Bloch NMR Flow Equation, Bessel Equation, Diffusion Equation,
Wave Equation, Schrödinger's Equation, Legendre's Equation,
Euler's Equation and Boubaker Polynomials.

2.1 Introduction

Advances in computers, mathematics, and science, is giving way to nonsurgical tools in the diagnosis of certain diseases. Besides X-ray imaging, now over 100 years old, the technologies include computed tomography (CT scans), positron-emission tomography (PET scans), ultrasound imaging, or sonography and magnetic resonance imaging (MRI).

Magnetic Resonance Imaging [1-34] uses a powerful magnetic field along with radio waves (not X-rays) and a computer to produce highly detailed "slice-by-slice" pictures of virtually all internal structures of the body. The results enable physicians to examine parts of the body in minute detail and identify disease in ways that are not possible with other techniques. For example, MRI is one of the few imaging tools that can see through bone, making it an excellent tool for examining the brain and other soft tissue.

Patients must remain still during the imaging process. And because the scan takes place as the patient slides through a rather small tunnel in the machine, some people experience claustrophobia. In recent times, though, open MRI scanners have been developed for patients who are anxious or obese. Naturally, no metal objects such as pens, watches, jewelry, hairpins, and metal zippers as well as credit cards and other magnetically sensitive items are allowed into the examination room.

If a contrast fluid is used, there is a slight risk of allergic reaction, but the risk is less than that associated with the iodine-based substances commonly used with X-rays and CT scans. Otherwise, MRI poses no known risk to the patient. However, because of the effect of the strong magnetic field, patients with certain surgical implants or metal fragments from injuries may be unable to have an MRI. So if an MRI is recommended, be sure to tell your doctor and your MRI technologist if you have any of these things. MRI does not use potentially harmful radiation, and it is particularly good at detecting tissue abnormalities, especially those that may be obscured by bone.

At present, the main thrust of research seems to be to improve technology that is already available. For example, researchers are developing MRI scanners that operate with a much weaker magnetic field than that of present devices, thus considerably reducing costs. A new technology under development is called molecular imaging (MI). Designed to detect changes within the body at the molecular level, MI promises very early detection and treatment of disease. MRI technology has reduced the need for many painful, risky, and even unneeded exploratory operations. And when imaging leads to early diagnosis and treatment of disease, the outcome may be much better. The equipment, however, is expensive—some machines costing well over a million dollars.

Despite over 50 years of the use of MRI for various investigations, the choice of technique parameters still relies to a great extent on experience. Research efforts to optimize the choice of parameter settings which yield sufficient image quality at the lowest possible cost are still rare. True optimization requires 1) estimation of the image quality needed to make a correct diagnosis and 2) methods to investigate all possible means of achieving this image quality in order to be able to decide which of them gives the lowest cost. Since the Bloch NMR equations are fundamental to all NMR/MRI computations, simulations and experiments, it can be fruitful, rewarding and beneficial with exciting results if these problems could be approached purely mathematically by solving the fundamental Bloch NMR equations analytically using all known mathematical techniques available both classical and quantum formulations. As such it presents significant challenge for the mathematical scientists, physicists, engineers and computer scientists to apply any of the fundamental and well known equations derived from the Bloch NMR flow equation as presented in this chapter to reveal most of the current unknowns but can enhance present understandings in the field of NMR/MRI.

Many of human diseases such as cancer, diabetes, arteriosclerosis and stroke, Alzheimer's disease, AIDS, etc, have all been known to be diseased conditions which take place at quantum (molecular) level. If we can see exactly what goes on at that level, we may have thorough understanding of their specific causes (or how they are caused), trace and monitor their progression and get the best cure for them. It is hoped that due to the ability of magnetic resonance to probe right to the fundamental level, we may be able to image human cellular functions and such imaging modalities would definitely help in the understanding of the human diseased conditions. Information gathered from the images can then be added to the present medical database to make it more comprehensive and thus permit the physician to make a more specific diagnosis, prognosis and possibly the appropriate therapy. The basic challenge in this direction is finding the right mathematical frameworks which appropriately describe the processes involved.

2.2 The Bloch NMR Equations

Magnetic resonance is a physical phenomenon whereby nuclei containing an odd number of particles, when in the presence of a magnetic field, absorb radio frequency waves at specific (resonance) frequencies. The magnitude of the radio frequency (RF) waves provides information about the molecules containing the nuclei. The nuclei have an intrinsic spin property, which generates a local magnetic field. The nuclei are also precessing around their axes with a velocity that is proportional to the strength of the external field (the Larmor equation). The nuclei are therefore often called spins. Magnetic resonance imaging (MRI) is a non-invasive technique used to obtain tomographic images of any desired plane of the body and by means of magnetic resonance velocity mapping; it is possible to quantify blood flow. In MRI, the spins or magnetic moments are exposed to a strong external magnetic field which will force the spins to line up in alignment with the field. In this state, the spins are at the lowest energy state and posses longitudinal magnetic properties. The magnetization at this point is the equilibrium magnetization M_0. Applying a radio frequency (RF) signal in a direction perpendicular to the spins at their resonance (Larmor) frequency causes the spin to absorb energy and hence tips the net magnetization vector of all the spins toward the transverse plane. This creates a net transverse magnetization vector. This vector will also precess about the external field and begin to relax towards alignment with the external field again (the lowest energy states). That is, the net magnetization moves back to align with the external field and hence we say that motion has occurred. This motion of the net magnetization is guided by a set of equations known as the Bloch equations which are the equations of motion for the net magnetization vector M of a sample of spins placed in a main magnetic field B_0 (where the components of M are M_x, M_y and M_z).

The behaviour of the transverse magnetization vector can be detected by the receiving unit in the scanner, the rf coil, and produce an rf signal. This rf signal is a fine wave at the Larmor frequency. The rate at which the net longitudinal magnetization vector builds up again to the equilibrium values is constant and is expressed by the T_1 relaxation time. The rate at which the transverse magnetization vector decreases is also constant and expressed by the T_2 relaxation time. T_1 and T_2 are the major parameters influencing the amplitude of the magnetic resonance signal. They depend on the molecular environment of the tissue and allow the distinction of different types of tissues.

The magnetic vector μ of a spinning, charged particle lies along the axis of rotation. The surrounding magnetic field symbolized by the vector H, exerts a torque that tends to bring μ and H into alignment. However, this torque also interacts with the angular momentum vector; the effect of this interaction is to cause the spin axis to describe a cone about the direction of the magnetic field. This phenomenon is called the Larmor precession, named after Sir Joseph Larmor, the Irish Physicist, who was the first to calculate the rate at which energy is radiated by an accelerated electron and the first to explain the splitting of spectrum lines by a magnetic field.

When the natural frequency of the precessing nuclear magnets corresponds to the frequency of a weak external radio wave striking the material, energy is absorbed from the radio wave. This selective absorption, called resonance, may be produced either by tuning the natural frequency of the nuclear magnets to that of a weak radio wave of fixed frequency or by tuning the frequency of the weak radio wave to that of nuclear magnets determined by the strong constant external magnetic field. This motion of the magnetization vector of uncoupled spins is easily expressed in terms of the Bloch NMR equations.

Almost all MRI concepts, dynamics and experiments are governed by the Bloch NMR equations. These equations relate the macroscopic model of magnetization to the applied radiofrequency, gradient and static magnetic fields. The dynamics of the changes in bodies containing NMR - sensitive nuclei, its physical changes (for example, freely diffusing or bound within a cavity) are carefully captured by the Bloch equation: a phenomenological equation describing the physics of magnetic moments – such as the moment of the water proton as a precessional gyroscopic motion in the presence of exponential damping (T_1 and T_2), perturbing magnetic fields (the fixed B_o, and the time - varying radiofrequency B_1).

The Bloch NMR equations are a set of coupled differential equations describing the behaviour of the macroscopic magnetization vector under any conditions. A form of the equations [35-41] is given as:

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \tag{2.1}$$

$$\frac{dM_y}{dt} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{2.2}$$

$$\frac{dM_z}{dt} = -\gamma M_z B_1(x) + \frac{M_o - M_z}{T_1} \tag{2.3}$$

The parameters are defined in the macroscopic frame of reference M_x, M_y (Transverse magnetization) and M_z (longitudinal magnetization) are magnetizations along x, y and z directions, M_o is the equilibrium magnetization (along the z direction), $B_1(x)$ is the Radiofrequency (RF) magnetic field which can be constant, depending on x and/or t. T_1 is the longitudinal or spin-lattice relaxation time, T_2 is the transverse or spin-spin relaxation time and γ is the gyro magnetic ratio of fluid spins.

The total magnetic field is given as:

$$\vec{B} = \vec{B}_o + \vec{B}_1(x) \tag{2.4}$$

where B_o is the static magnetic field. All these parameters, as may be related to MRI will be discussed in full detail in section.

Since the MRI spin are always in motion, they must be treated with reference to their dynamics. The features of this dynamics are very much pronounced in fluids especially in biological systems.

From the kinematic theory of moving fluids, given a property M of the fluid, then the rate at which this property changes with respect to a point moving along with the fluid will be the total derivative:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}\frac{dx}{dt} + \frac{\partial M}{\partial y}\frac{dy}{dt} + \frac{\partial M}{\partial z}\frac{dz}{dt} \tag{2.5}$$

where $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are the components of the fluid velocity $\vec{v}$. The change in the parameter, *dM*, occurring during the time *dt*, at the position of a moving fluid particle which moves from *x, y, z* to *x+dx, y+dy, z+dz* during this time, will be:

$$dM = M(x+dx, y+dy, z+dz, t+dt) - M(x,y,z,t)$$

$$dM = \frac{\partial M}{\partial t}dt + \frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy + \frac{\partial M}{\partial z}dz$$

equation (2.5) is obtained if $dt \to 0$. We can also write this equation in the form:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}v_x + \frac{\partial M}{\partial y}v_y + \frac{\partial M}{\partial z}v_z \tag{2.6}$$

and

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \vec{v} \cdot \nabla M \tag{2.7}$$

where the second expression is shorthand for the first, in accordance with the conventions for using the symbol ∇. The total derivative $\frac{\mathrm{dM}}{\mathrm{dt}}$ is also a function of *x, y, z,* and *t*. A similar relation holds between partial and total derivative of any quantity, and we may write, symbolically,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla$$

where *v* is the fluid velocity and $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

The Bloch equations become:

$$\frac{dM_x}{dt} = \frac{\partial M_x}{\partial t} + v \cdot \nabla M_x = -\frac{M_x}{T_2} \tag{2.8}$$

$$\frac{dM_y}{dt} = \frac{\partial M_y}{\partial t} + v \cdot \nabla M_y = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{2.9}$$

$$\frac{dM_z}{dt} = \frac{\partial M_z}{\partial t} + v \cdot \nabla M_z = -\gamma M_z B_1(x) + \frac{M_o - M_z}{T_1} \tag{2.10}$$

Considering fluid flow along horizontal x – direction, partial derivatives along the y and z directions are ignored. Therefore:

$$v \cdot \nabla M_x = v \frac{\partial M_x}{\partial x}$$

Similarly, $v \cdot \nabla M_y = v \dfrac{\partial M_y}{\partial x}$ and $v \cdot \nabla M_z = v \dfrac{\partial M_z}{\partial x}$

Equations (2.8 - 2.10) then become:

$$\frac{dM_x}{dt} = \frac{\partial M_x}{\partial t} + v \frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \tag{2.11}$$

$$\frac{dM_y}{dt} = \frac{\partial M_y}{\partial t} + v \frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{2.12}$$

$$\frac{dM_z}{dt} = \frac{\partial M_z}{\partial t} + v \frac{\partial M_z}{\partial x} = -\gamma M_z B_1(x) + \frac{M_o - M_z}{T_1} \tag{2.13}$$

2.3 The General Bloch NMR Flow Equation

The Bloch NMR flow equations can be written as:

$$\frac{\partial M_x}{\partial t} + v \frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \tag{2.14}$$

$$\frac{\partial M_y}{\partial t} + v \frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{2.15}$$

$$\frac{\partial M_z}{\partial t}+v\frac{\partial M_z}{\partial x}=-\gamma M_z B_1(x)+\frac{M_o-M_z}{T_1} \tag{2.16}$$

From equation (2.16), we have:

$$\mathrm{v}\frac{\partial \mathrm{M_z}}{\partial \mathrm{x}}+\frac{\partial \mathrm{M_z}}{\partial \mathrm{t}}+\frac{\mathrm{M_z}}{\mathrm{T_1}}=-\gamma \mathrm{M_y B_1}(\mathrm{x})+\frac{\mathrm{M_o}}{\mathrm{T_1}}$$

or

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)M_z=-\gamma M_y B_1(x)+\frac{M_o}{T_1}$$

$$\mathrm{M_z}=\left(-\gamma \mathrm{M_y B_1}(\mathrm{x})+\frac{\mathrm{M_o}}{\mathrm{T_1}}\right)\frac{1}{\left(\mathrm{v}\frac{\partial}{\partial \mathrm{x}}+\frac{\partial}{\partial \mathrm{t}}+\frac{1}{\mathrm{T_1}}\right)} \tag{2.17}$$

Substituting for M_z in equation (2.15) gives:

$$\frac{\partial \mathrm{M_y}}{\partial \mathrm{t}}+\mathrm{v}\frac{\partial \mathrm{M_y}}{\partial \mathrm{x}}$$
$$=\gamma\left(-\gamma \mathrm{M_y B_1}(\mathrm{x})+\frac{\mathrm{M_o}}{\mathrm{T_1}}\right)\frac{1}{\left(\mathrm{v}\frac{\partial}{\partial \mathrm{x}}+\frac{\partial}{\partial \mathrm{t}}+\frac{1}{\mathrm{T_1}}\right)_z}\mathrm{B_1}(\mathrm{x})-\frac{\mathrm{M_y}}{\mathrm{T_2}}$$

$$\begin{aligned}&\mathrm{v}\frac{\partial \mathrm{M_y}}{\partial \mathrm{x}}\left(\mathrm{v}\frac{\partial}{\partial \mathrm{x}}+\frac{\partial}{\partial \mathrm{t}}+\frac{1}{\mathrm{T_1}}\right)+\frac{\partial \mathrm{M_y}}{\partial \mathrm{t}}\left(\mathrm{v}\frac{\partial}{\partial \mathrm{x}}+\frac{\partial}{\partial \mathrm{t}}+\frac{1}{\mathrm{T_1}}\right)+\frac{\mathrm{M_y}}{\mathrm{T_2}}\left(\mathrm{v}\frac{\partial}{\partial \mathrm{x}}+\frac{\partial}{\partial \mathrm{t}}+\frac{1}{\mathrm{T_1}}\right)\\&=\gamma\left(-\gamma \mathrm{M_y B_1}(\mathrm{x})+\frac{\mathrm{M_o}}{\mathrm{T_1}}\right)\mathrm{B_1}(\mathrm{x})\end{aligned} \tag{2.18a}$$

For general pulsed NMR/MRI experiment $B_1(x)$ in equation (2.18a) will be replaced by $B_1(x,t)$. This is valid even ia a rotating frame. Equation (2.18a) can then be written in a more general form as [38]:

$$v^2\frac{\partial^2 M_y}{\partial x^2}+v\frac{\partial^2 M_y}{\partial x\partial t}+\frac{v}{T_1}\frac{\partial M_y}{\partial x}+v\frac{\partial^2 M_y}{\partial x\partial t}+\frac{\partial^2 M_y}{\partial t^2}+\frac{1}{T_1}\frac{\partial M_y}{\partial t}$$
$$+\frac{v}{T_2}\frac{\partial M_y}{\partial x}+\frac{1}{T_2}\frac{\partial M_y}{\partial t}+\frac{1}{T_1T_2}M_y$$
$$=-\gamma^2B_1^{\ 2}(x,t)M_y+\frac{\gamma B_1(x,t)M_o}{T_1}$$

$$v^2\frac{\partial M_y}{\partial x^2}+2v\frac{\partial^2 M_y}{\partial x\partial t}+v\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial x}$$
$$+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial t}+\frac{\partial^2 M_y}{\partial t^2}+\left(\frac{1}{T_1T_2}+\gamma^2B_1^{\ 2}(x,t)\right)M_y \tag{2.18b}$$
$$=\frac{\gamma B_1(x,t)M_o}{T_1}$$

Equation (2.18b) is a general second order differential equation which can be applied to any fluid flow problem. At any given time t, we can obtain information about the system, provided that appropriate boundary conditions are applied. From equation (2.18b), we can obtain the diffusion equation, the wave equation, telephone and telegraph equations e.t.c, and solve them in terms of NMR parameters by the application of appropriate initial or boundary conditions. Hence, we could get very important information about the dynamics of the system. It should be noted however that the term $F_0\gamma B_1(x,t)$ is the forcing function $(F_0 = M_0/T_1)$. If the function is zero, we have a freely vibrating system; else, the system is undergoing a forced vibration.

2.4 The Time - Independent Bloch NMR Flow Equation

For a steady flow, all partial derivatives with respect to time can be set to zero (time independent). Hence equations (2.11-2.13) become:

$$v\frac{dM_x}{dx} = -\frac{M_x}{T_2} \tag{2.19}$$

$$v\frac{dM_y}{dx} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{2.20}$$

$$v\frac{dM_z}{dx} = -\gamma M_z B_1(x) + \frac{M_o - M_z}{T_1} \tag{2.21}$$

From equation (2.21) we write:

$$v\frac{dM_z}{dx} = -\gamma M_y B_1(x) + \frac{M_o}{T_1} - \frac{M_z}{T_1} \tag{2.22}$$

collecting the like term in equation (2.22) gives:

$$\left(v\frac{d}{dx} + \frac{1}{T_1}\right)M_z = -\gamma M_y B_1(x) + \frac{M_o}{T_1} \tag{2.23}$$

From equation (2.20, 2.21 and 2.23) we have,

$$v\frac{dM_y}{dx}\left(v\frac{d}{dx} + \frac{1}{T_1}\right) + \frac{M_y}{T_2}\left(v\frac{d}{dx} + \frac{1}{T_1}\right) = -\gamma^2 M_y B_1^{\ 2}(x) + \frac{\gamma B_1(x) M_o}{T_1} \tag{2.24}$$

$$v^2\frac{d^2M_y}{dx^2} + v\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{dM_y}{dx} + \left(\frac{1}{T_1T_2} + \gamma^2 B_1^{\ 2}(x)\right)M_y = \frac{\gamma B_1(x) M_o}{T_1}$$

$$v^2\frac{d^2M_y}{dx^2} + vT_o\frac{dM_y}{dx} + \left(T_g + \gamma^2 B_1^{\ 2}(x)\right)M_y = \frac{M_o}{T_1}\gamma B_1(x) \tag{2.25}$$

Equation (2.25) is a time independent Bloch NMR flow equation [39-46].

2.5 The Time - Dependent Bloch NMR Flow Equation

For a flow that is independent of the space coordinate, *x*, that is, the magnetization does not change appreciably over a large *x* for a very long time, then all partial derivatives with respect to *x* could be set to zero (time dependent) [38]. From equation (2.3) we write:

$$\frac{dM_z}{dt} = -\gamma M_y B_1(t) + \frac{M_o}{T_1} - \frac{M_z}{T_1} \tag{2.26}$$

$$\left(\frac{d}{dt} + \frac{1}{T_1}\right) M_z = -\gamma M_y B_1(t) + \frac{M_o}{T_1} \tag{2.27}$$

Substituting for M_z in equation (2.27) into equation (2.26) gives:

$$\frac{dM_y}{dt}\left(\frac{d}{dt} + \frac{1}{T_1}\right) + \frac{M_y}{T_2}\left(\frac{d}{dt} + \frac{1}{T_1}\right) = -\gamma^2 B_1^{\,2}(t) M_y + \frac{\gamma B_1(t) M_o}{T_1}$$

$$\frac{d^2M_y}{dt^2} + T_o \frac{dM_y}{dt} + \left(T_g + \gamma^2 B_1^{\,2}(t)\right) M_y = \frac{M_o}{T_1}\gamma B_1(t)$$

$$\frac{d^2M_y}{dt^2} + \frac{1}{T_1}\frac{dM_y}{dt} + \frac{1}{T_2}\frac{dM_y}{dt} + \frac{1}{T_1 T_2} M_y + \gamma^2 B_1^{\,2}(t) M_y = \frac{\gamma B_1(t) M_o}{T_1} \tag{2.28}$$

Equations (2.18b, 2.25, 2.28) are fundamental equations that can appropriately guide the generation of MRI signal of any kind in any coordinate. This is possible because these equations can easily be transformed to known equations commonly used in Mathematics, Physics and Engineering to solve real life problems. Some of the equations will be derived in the next sections.

2.6 Diffusion MRI Equation

Starting from equation (2.18b), we can assume a solution of the form:

$$M_y(x,t) = Ae^{\mu x + \eta t} \tag{2.29}$$

subject to the following theoretical conditions (the limiting case of non adiabatic small rf limit):

$$\gamma^2 B_1^{\ 2}(x,t) << \frac{1}{T_1 T_2} \tag{2.30}$$

where μ and η are dependent on the NMR parameters and B_1 is independent of x and t.

Taking $\eta^2 = T_g$ and $2\eta = T_o$.

Equation (2.18b) becomes:

$$v^2 \frac{\partial^2 M_y}{\partial x^2} + T_o \frac{\partial M_y}{\partial t} = F_o \gamma B_1(x,t) \tag{2.31}$$

If we write

$$D = -\frac{v^2}{T_o} \tag{2.32}$$

Then equation (2.31) becomes:

$$\frac{\partial M_y}{\partial t} = D \frac{\partial^2 M_y}{\partial x^2} + F_o \gamma B_1(x,t) \tag{2.33}$$

This can be written in generalized co-ordinate as [47-55]:

$$\frac{\partial M_y}{\partial t} = D \nabla^2 M_y + F_o \gamma B_1(x,t) \tag{2.34}$$

If *D* represents the diffusion coefficient, then Equation (2.34) is the equation of diffusion of magnetization as the nuclear spins move. The function

$F_o \gamma B_1(x,t)$ is the forcing function, which shows that the application of the rf B_1 field has an influence on the diffusion of magnetization within a voxel. It is interesting to note that the dimension of Equation (2.33) exactly matches that of diffusion coefficient.

Equation (2.34) is only applicable when *D* in non – directional. That is, we have a constant diffusion coefficient (isotropic medium). In a later section equation (2.34) will be considered for restricted diffusion in various geometries.

This model would work quite well for molecules that move very short distances over a very considerable amount of time.

where

$$\mathrm{F_o} = \frac{\mathrm{M_o}}{\mathrm{T_1}};\ \mathrm{T_g} = \frac{1}{\mathrm{T_1 T_2}} \text{ and } \mathrm{T_0} = \frac{1}{\mathrm{T_1}} + \frac{1}{\mathrm{T_2}}$$

γ is the gyromagnetic ratio, *D* is the diffusion coefficient, v is the fluid velocity, T_1 is the spin lattice relaxation time, T_2 is the spin relaxation time, M_o is the equilibrium magnetization, $B_1(x, t)$ is the applied magnetic field and M_y is the transverse magnetization. Solutions to equation (2.1) have been discussed by a number of analytical methods [12, 21], and for the present purpose it is sufficient to design the NMR system in such a way that the transverse magnetization M_y, takes the form of a plane wave,

2.7 Wave MRI Equation

Based on equations (2.29), we can write equation (2.18b) in the form wave equation:

$$v^2 \frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_y}{\partial t^2} = F_o \gamma B_1(x,t) \tag{2.35}$$

Equation (2.35) only holds when:

$$\eta T_o = -\frac{1}{T_1 T_2} \quad , \quad 2\eta = -T_o \tag{2.36}$$

In three dimensions equation (2.35) becomes:

$$v^2 \nabla^2 M_y + \frac{\partial^2 M_y}{\partial t^2} = F_o \gamma B_1(r,t) \tag{2.37}$$

In the spherical polar geometries, we can write equation (2.37) as:

$$v^2 \left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2} \right) + \frac{\partial^2 M_y}{\partial t^2} = F_o \gamma B_1(r,t) \tag{2.38}$$

When the rf B_1 field is at its peak, it is expected that the angle between the initial position and the resulting one is π. If the transverse magnetization is radially symmetric, we can write:

$$v^2 \left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} \right) + \frac{\partial^2 M_y}{\partial t^2} = F_o \gamma B_1(r,t) \tag{2.39}$$

2.8 The Bessel Equation

We study the flow properties of the modified time independent Bloch NMR flow equations which describes the dynamics of the hydrogen atom under the influence of rf magnetic field as follows [1-10]:

$$v^2 \frac{d^2 M_y}{dx^2} + T_0 v \frac{dM_y}{dx} + S(x) M_y = \frac{M_0 \gamma B_1(x)}{T_1} \tag{2.40}$$

where $S(x) = \gamma^2 B_1^2(x) + T_g$, $T_g = \frac{1}{T_1 T_2}$, $T_0 = \frac{1}{T_1} + \frac{1}{T_2}$.

In equation (2.25), the spin velocity ν is constant and distance x can be defined as:

$$\nu = xT_o = \frac{x}{\tau} \text{ and } T_o = \frac{1}{\tau} \tag{2.41}$$

where T_o is the T1 an T2 relaxation rates of the spins which may be changing from pixel to pixel within the distance x. If the MRI signal is sampled when the applied radiofrequency energy successfully displaces most of the spin unto the transverse plane ($M_0 \approx 0$), equation (2.40) then becomes:

$$x^2 \frac{d^2 M_y}{dx^2} + T_o^2 x \frac{dM_y}{dx} + \left((k^2 x^2 + \beta^2\right) M_y = 0 \tag{2.42}$$

where

$$\gamma B_1(x) = \gamma G x \tag{2.43a}$$

$$\mathrm{k} = \gamma \mathrm{GT_o}$$

and

$$\beta^2 = \frac{T_o^2}{T_1 T_2} \tag{2.43b}$$

Equation (2.42) is an equation transformable to Bessel function. When there is no gradient G, all the spins are rotating at the same frequency and therefore, in the rotating frame of reference at rotating at ω_o, the spins appears stationary with no phase difference. In the presence of gradient G, the spins start presecing with slightly difference frequencies winding them into a helix and hence a phase difference is induced. As the gradient strength and/or duration τ, increases, the pith of a helix become becomes smaller resulting in a smaller wavelength and correspondingly a higher k value.

The equation for the total MRI signal from a slice in the x, y, pane is:

$$S(k_x, k_y) = \int \int f(x, y) e^{-i(k_x x + k_y y)} dxdy \tag{2.44a}$$

where

$$k_y = \gamma G_y T_o \tag{2.44b}$$

$$k_x = \gamma G_x T_o \tag{2.44c}$$

Equation (2.44) is the fundamental equation for MRI. It gives detail information on MRI signal within a voxel. f(x, y) is the distribution of the MRI signal over the slice d, at the time just after the excitation. Equations (2.44b, 2.44c) are the k- values along the phase and frequency encoding axes respectively. The Fourier transform of equation (2.44) is:

$$f(x, y) = \int \int S(k_x, k_y) e^{-i(k_x x + k_y y)} dk_x k_y \tag{2.45}$$

2.9 The NMR Schrodinger Wave Equation

NMR is a quantum phenomenon and like all other quantum phenomena is best described by quantum mechanics. It will be enormously valuable if quantum mechanics as a tool for understanding the NMR microscopic nature is developed in parallel with the growth of NMR Physics. It is our goal to develop the Bloch NMR flow equations in terms of quantum mechanical wave functions which can predict analytically and precisely the probability of events or outcome.

In equation (2.25) It is convenient to use as dependent variable the departure of the stream function from its classical form and write:

$$M_y = \psi(x)e^{\lambda x} \tag{2.46}$$

where $\lambda = \frac{-1}{2vT_O}$, ν is the instantaneous velocity of the fluid and ψ(x) is a special function of the transverse magnetization M_y, which depends on the dynamical state of the fluid particle. When M_y is maximum and M_o is minimum (say $M_o = 0$). For a maximum value of M_y (when $M_o = 0$) we can write equation (2.40) as:

$$\frac{d^2\psi}{dx^2} + \frac{\gamma^2 B_1^2}{v^2}\psi = 0 \tag{2.47a}$$

subject to the following conditions:

(i) $$e^{\lambda x} \neq 0 \tag{2.47b}$$

(ii) Resonance condition exists at Larmor frequency

$$f_o = \gamma B - \omega = 0$$

(iii) $\gamma^2 B_1^2(x) >> \left(T_g - T_R\right)$

where $T_g = \frac{1}{T_1T_2}$, $T_R = \frac{1}{4T_o^2}$ and $\frac{1}{T_o} = \frac{1}{T_1} + \frac{1}{T_2}$.

γ denotes the gyromagnetic ratio of fluid spins; ω/2π is the rf excitation frequency; fo/γ is the off- resonance field in the rotating frame of reference. T1 and T2 are the spin-lattice and spin-spin relaxation times respectively, the reciprocals of T1 and T2 are defined as relaxation rates. rf B1 is treated as constant and of the order of 1G. The exponential function in equation (2.47b) can be defined as follows,

$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = F(x) \tag{2.48}$$

Equation (2.48) is extremely useful in obtaining approximations to complicated formulas, valid when x is small. In particular, when:

$$x = 4vT_o \tag{2.49}$$

Equation (2.47c) becomes:

$$F(x) = 1$$

Equation (2.27a) becomes the Schrödinger wave equation in 1-D given by [40]:

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2}(E - E_p(x))\psi = 0 \tag{2.50}$$

where

$$\frac{\gamma^2 B_1^2(x)}{v^2} = \frac{2\mu}{\hbar^2}[E - E_p(x)] \tag{2.51}$$

E and $E_p(x)$ are the total energy and potential energy of the fluid particle respectively. Equation (2.48) can easily be solved if E_p is constant, with a solution of the form $e^{\pm ikx}$. But if E_p varies with x, one may find solution in the form:

$$\psi(x) = e^{iw(x)} \tag{2.52}$$

To simplify this problem,

$$\text{Let } k(x) = \left\{\frac{2\mu}{\hbar^2}(E - v)\right\}^{\frac{1}{2}} \tag{2.53}$$

Substituting equation (2.52) into (2.51) gives equation for the x-dependent phase.

Hence, w(x) satisfies:

$$i\frac{d^2w}{dx^2}-\left(\frac{dw}{dx}\right)^2+[k(x)]^2=0 \tag{2.54}$$

Note that equations (2.54) and (2.51) are equivalent.

For a free particle, $\frac{d^2w}{dx^2}=0$. Hence we can neglect the second derivative term $\frac{d^2w}{dx^2}$ in equation (2.54) and this will lead to our first approximationw_0 in w.

$$\begin{aligned}
&\left(\frac{dw_0}{dx}\right)^2=\left[k(x)\right]^2\\
&w'^2_0=\left[k(x)\right]^2\\
&w'_0=k(x)\\
&w_0=\pm\int^x k(x)dx+C
\end{aligned} \tag{2.55}$$

Equation (2.55) is the approximation to the wave function.

Setting up a successive approximation; from equation (2.54), we can write:

$$\left(\frac{dw}{dx}\right)^2=i\frac{d^2w}{dx^2}+[k(x)]^2 \tag{2.56}$$

By substituting the n^{th} approximation on the R.H.S, we obtain the $(n+1)^{th}$ approximation by quadrature.

$$w_{n+1}=\pm\int^x\sqrt{[k(x)]^2+iw''_n(x)}dx+C_{n+1} \tag{2.57}$$

Thus, for n=0, we obtain:

$$\begin{aligned} w_1 &= \pm\int^{x} \sqrt{[k(x)]^2 + iw_0''(x)}dx + C_1 \\ w_1 &= \pm\int^{x} \sqrt{[k(x)]^2 + ik'(x)}dx + C_1 \end{aligned} \tag{2.58}$$

It is expected that w_1 be close to w_0, for this approximation to approximate the wave function.

Hence

$$|k'(x)| << |k^2(x)| \tag{2.59}$$

If condition (2.59) holds, one may expand the integrand in equation (58) and obtain:

$$\begin{aligned} w_1(x) &\cong \pm\int \left[k(x) + \frac{i}{2}\frac{k'(x)}{k(x)} \right] dx + C_1 \\ w_1(x) &\cong \pm\int^{x} k(x)dx + \frac{i}{2}\log k(x) + C_1 \end{aligned} \tag{2.60}$$

The constant of integration only affects the normalization of $\Psi(x)$. It can be neglected until the desired approximation is made.

Hence the approximation in equation (2.54-2.60), called WKB approximation, leads to the approximate wave function.

$$\psi(x) = \frac{1}{\sqrt{k(x)}} e^{\pm i\int^{x} k(x)dx} \tag{2.61}$$

Taking k(x) as the effective wave number, we can define our wavelength as $\lambda(x) = \frac{2\pi}{k(x)}$.

Therefore condition (2.59) can be re-written as:

$$\lambda(x)\left|\frac{dp}{dx}\right| << |p(x)| \tag{2.62}$$

Consider a turning point and assuming that except in its immediate neighbourhood, WKB approximation is applicable. Changing the dependent and independent variables, we write:

$$u(x) = \sqrt{k(x)}\psi(x) \tag{2.63}$$

And

$$y = \int^{x} k(x)dx \tag{2.64}$$

By manipulating, we obtain:

$$\frac{d^2u}{dy^2} + \left[\frac{1}{4k^2}\left(\frac{dk}{dy}\right)^2 - \frac{1}{2k}\frac{d^2k}{dy^2} + 1\right]u = 0 \tag{2.65}$$

Substituting the particular value of k(x) given by equation (2.63), the integral of equation (2.64) can be evaluated, and choosing the lower limit of the integration as 0, we obtained,

$$y = \frac{\sqrt{2\mu}}{\sqrt{\hbar^2}\sqrt{E}}\left(\sqrt{E}\cdot\sqrt{\frac{4\pi\varepsilon_0 Ex^2 + ze^2 x}{4\pi\varepsilon_0}} + \frac{ze^2}{4\pi\varepsilon_0}\,\mathrm{In}\left\{\frac{\sqrt{4\pi\varepsilon_0 Ex} + \sqrt{4\pi\varepsilon_0 Ex + ze^2}}{\sqrt{ze^2}}\right\}\right) \tag{2.66}$$

where y is the measure of the distance from the classical turning point. Hence y is small near the turning point assuming the two limits of integration are close to each other. At points very far to the left or right from the turning point, WKB is applicable.

Expressing y in terms of k, and finding its derivative, equation (2.65) becomes:

$$\frac{d^2u}{dy^2}+\left(1+\frac{5}{36y^2}\right)u=0 \tag{2.67}$$

Let us attempt the solution of equation (2.67) in the form:

$$u(y)=y^{\lambda}\int e^{yt} f(t)\cdot dt \tag{2.68}$$

Substituting (2.68) into (2.67) gives,

$$\int\left[\lambda(\lambda-1)+2\lambda yt+y^2t^2+y^2+\frac{5}{36}\right]e^{yt}f(t)\cdot dt=0 \tag{2.69}$$

Choosing λ such that the terms which are constant in y vanish, it is required that:

$$\lambda(\lambda-1)+\frac{5}{36}=0 \tag{2.70a}$$

$$\lambda=\frac{1}{6},\frac{5}{36} \tag{2.70b}$$

The remaining expression in equation (2.69) is:

$$\int f(t)\left[2\lambda t+\left(1+t^2\right)\frac{d}{dt}\right]e^{yt}\cdot dt=0 \tag{2.71}$$

Integrating by part,

$$\int f(t)\left[2\lambda t+\left(1+t^2\right)\frac{d}{dt}\right]e^{yt}\cdot dt=0 \tag{2.72}$$

$$\int\left\{2\lambda tf(t)-\frac{d}{dt}\left[\left(1+t^2\right)f(t)\right]\right\}e^{yt}dt+\int\frac{d}{dt}\left[\left(1+t^2\right)f(t)e^{yt}\right]=0 \tag{2.73}$$

To successfully construct a solution of the proposed form in equation (2.68), the integrand in the first integral should be made to vanish and the path of integration is chosen so that the second integral disappears.

We therefore require:

$$2\lambda tf(t)=\frac{d}{dt}\left[\left(1+t^2\right)f(t)\right]$$

$$f(t)=f(0)\left(1+t^2\right)^{\lambda-1}$$

Using equation (2.70), we can write the general form of (2.67) as:

$$\frac{d^2u}{dy^2}+\left(1+\frac{\lambda(\lambda-1)}{y^2}\right)u=0 \tag{2.74}$$

It should be noted that if u_λ is a solution, $u_{1-\lambda}$ is also a solution of equation (2.74).

In deriving the WKB connection formulas,

$$u_\lambda^+(y)=y^\lambda\int_i^{-i\infty}\frac{e^{yt}}{\left(1+t^2\right)^{1-\lambda}}dt \tag{2.75}$$

and

$$u_\lambda^-(y)=y^\lambda\int_{-i}^{-i\infty}\frac{e^{yt}}{\left(1+t^2\right)^{1-\lambda}}dt \tag{2.76}$$

Recall that λis not an integer, this implies that $t = \pm i$ are branch points of the function $(1 + t^2)^{1-\lambda}$.

The asymptotic expansion of u_λ^+ and u_λ^- are needed in the WKB expansion for large imaginary values of y. Hence, we substitute in $u_\lambda^\pm$:

$$t = \pm i - \frac{z}{y} \tag{2.77}$$

and obtain:

$$u_\lambda^\pm(y) = -y^{\lambda-1} e^{\pm iy} \int_0^\infty \left(\frac{z^2}{y^2} \mp 2i\frac{z}{y} \right)^{\lambda-1} e^{-z} dz \tag{2.78}$$

For |y|large enough, a reasonable approximation to the asymptotic expansion of u is obtained by expanding the parenthesis in powers of $\frac{z}{y}$, then integrate term by term,

$$u_\lambda^+(y) \approx \mp 2^{\lambda-1} i\Gamma(\lambda) e^{\pm iy - i\lambda\left(\pi \mp \frac{\pi}{2}\right)} \tag{2.79}$$

Since $\left(\pi \mp \frac{\pi}{2}\right)$ is negative imaginary, the form of this solution is in agreement with the WKB approximation in the region of negative kinetic energy.

When the variable y is negative real, a different integration path is taken; we will choose a different limit of integration.

$$u_\lambda(y) = y^\lambda \int_{-i}^{+i} \frac{e^{yt}}{\left(1 + t^2\right)^{1-\lambda}} dt \tag{2.80}$$

and

$$u_\lambda(y) \approx 2^\lambda i\Gamma(\lambda)e^{i\pi\lambda} \cos\left(y + \lambda\frac{\pi}{2}\right). \tag{2.81}$$

This solution also agrees with the WKB approximation in the region of positive kinetic energy. Unless λ is half-integral, $u_\lambda(y)$ and $u_{1-\lambda}(y)$ are two linearly independent solution.

From equations (2.75), (2.76) and (2.80), we can define:

$$u_\lambda(y) = u_\lambda^-(y) - u_\lambda^+(y) \tag{2.82}$$

and

$$u_{1-\lambda}(y) = u_{1-\lambda}^-(y) - u_{1-\lambda}^+(y) \tag{2.83}$$

Near the turning point $y = 0$, the integral in equation (2.80) gives:

$$u_\lambda(y) = i\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)}{\Gamma\left(\lambda + \frac{1}{2}\right)} y^\lambda \left[1 + 0\left(y^2\right)\right] \tag{2.84}$$

This proves that the wave function in equations (2.46, 2.47, 2.50):

$$\psi = \frac{u}{\sqrt{k}} \propto \frac{u}{y^{\frac{1}{6}}} \tag{2.85}$$

Perfectly behave well near the turning point.

2.10 Time - Dependent NMR Schrodinger Equaion

At the molecular level the diffusion coefficient of a fluid particle is defined as:

$$D = -\frac{\hbar}{2im} \tag{2.86}$$

Substituting equation (2.86) into equation (2.33) gives the time-dependent NMR Schrodinger equation:

$$i\hbar\frac{\partial M_y}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 M_y}{\partial x^2} + \frac{i\hbar F_o}{T_o}\gamma B_1(x,t) \tag{2.87}$$

We can represent the transverse magnetization M_y as the propagation of a plane harmonic wave in the x-direction in the form of equation (2.29) and write:

$$M_y(x,t) = Ae^{i(kx-\omega t)} \tag{2.88}$$

where A is a constant, μ = ik and η = -iω. Equation (2.88) represents a typical propagating matter wave where k, measures the wave vector and ω, the angular frequency of the wave. Splitting the transverse magnetization into its space and time parts in the form,

$$M_y(x,t) = X(x)T(t) \tag{2.89}$$

we can write

$$X(x) = A_1 e^{ikx} \tag{2.90}$$

$$T(t) = A_2 e^{-i\omega t} \tag{2.91}$$

A wave in the x-t space propagates by joint oscillations represented by equations (2.90-2.91) each of them is capable of exciting the other. By differentiating equation (2.90) twice and making use of:

$$E = \frac{k^2\hbar^2}{2m} + E_p(x) \tag{2.92}$$

we arrive at the Schrodinger's time-independent wave equation, expressed as in equation (2.50):

$$-\frac{\hbar^2}{2m}\frac{d^2X(x)}{dx^2} = [E - E_p(x)]X(x) \tag{2.93}$$

where E and $E_p(x)$ denote the total and potential energies of the particle respectively. Similarly, differentiating equation (2.91) once with respect to t and making use of the relation:

$$E = \hbar\omega$$

we arrive at,

$$-i\hbar\frac{dT(t)}{dt} = ET(t) \tag{2.94}$$

On combining equations (2.93) and (2.94) one arrives at the Schrodinger's time-dependent equation.

$$\hat{H}M_y(x,t) = i\hbar\frac{\partial}{\partial t}M_y(x,t) \tag{2.95}$$

where the $\hat{H}$ operator stand for:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + E_p(x) \tag{2.96}$$

Equation (2.96) is well-known as the Hamiltonian of the particle. It should be mentioned that equation (2.95) is the true equation representing the motion of microscopic particles through a given space. This is applicable even during a quantum measurement. As long as matter exhibits wave-particle dualism, equations (2.93) and (2.94) are both valid and their solutions are readily obtainable by solving them. The most general wave function may then be obtained by forming a suitable product of X(x) and T(t).

2.11 NMR Legendre Equation and Boubaker Polynomial

When M_y is maximum and M_o is minimum (say $M_o = 0$), we can write equation (2.25) as:

$$\frac{d^2 M_y}{dx^2} + \frac{T_0}{v}\frac{dM_y}{dx} + \frac{S(x)}{v^2} M_y = 0 \tag{2.97}$$

Where $T_0 = \frac{1}{T_1} + \frac{1}{T_2}$, $S(x) = \gamma^2 B_1^2(x) + \frac{1}{T_1 T_2}$.

If we then write that:

$$\frac{T_0}{v} = \frac{1}{l}\cot\frac{x}{l} \tag{2.98}$$

$$\frac{S(x)}{v^2} = \frac{1}{l^2} n(n+1) \tag{2.99}$$

The small rf limiting condition:

$$\frac{1}{T_1 T_2} >> \gamma^2 B_1^2(x)$$

gives

$$\frac{1}{T_1 T_2 v^2} = \frac{n(n+1)}{l^2}$$

$$\frac{l^2}{T_1 T_2 v^2} = n(n+1)$$

where l is a parameter in length or any other unit of distance. It is worthy of note that equation (2.97) is obtainable from the expression:

$$\frac{d^2M_y}{dx^2}+\frac{T_0}{v}\frac{dM_y}{dx}+\frac{S(x)}{v^2}M_y=\frac{M_0}{T_1}\gamma B_1(x) \tag{2.100}$$

Under two conditions

1. When the rf $B_1(x)$ field applied, has a maximum value, so that M_y is maximum; and $M_0 \approx 0$.

2. When the rf $B_1(x)$ field is just removed (so that $\gamma B_1(x) = 0$).

However, condition 1 seems to favour most part of this particular write-up.

Equation (2.97) can then be written as:

$$\begin{aligned}&\frac{d^2M_y}{dx^2}+\frac{1}{l}\cot\frac{x}{l}\frac{dM_y}{dx}+\frac{1}{l^2}n(n+1)M_y=0\\&\frac{d^2M_y}{dx^2}+\frac{1}{l}\frac{\cos\frac{x}{l}}{\sin\frac{x}{l}}\frac{dM_y}{dx}+\frac{1}{l^2}n(n+1)M_y=0\end{aligned} \tag{2.101}$$

Multiplying equation (2.101) all through by $sin\frac{x}{l}$, it follows that:

$$\sin\frac{x}{l}\frac{d^2M_y}{dx^2}+\frac{1}{l}\cos\frac{x}{l}\frac{dM_y}{dx}+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0 \tag{2.102}$$

It would be noted that:

$$\sin\frac{x}{l}\frac{d^2M_y}{dx^2}+\frac{1}{l}\cos\frac{x}{l}\frac{dM_y}{dx}\equiv\frac{d}{dx}\left(\sin\frac{x}{l}\frac{dM_y}{dx}\right) \tag{2.103}$$

Hence, equation (2.101) becomes:

$$\frac{d}{dx}\left(\sin\frac{x}{l}\frac{dM_y}{dx}\right)+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0 \tag{2.104}$$

If we define:

$$\varsigma=\cos\frac{x}{l} \tag{2.105}$$

and

$$\frac{dM_y}{dx}=\frac{dM_y}{d\varsigma}\cdot\frac{d\varsigma}{dx}=-\frac{1}{l}\sin\frac{x}{l}\frac{dM_y}{d\varsigma}$$

$$\sin\frac{x}{l}\frac{dM_y}{dx}=-\frac{1}{l}\sin^2\frac{x}{l}\frac{dM_y}{d\varsigma}$$

but

$$-\sin^2\frac{x}{l}=\cos^2\frac{x}{l}-1=\varsigma^2-1 \tag{2.106}$$

Therefore, equation (2.106) becomes:

$$\frac{d}{dx}\left\{\frac{\left(\varsigma^2-1\right)}{l}\frac{dM_y}{d\varsigma}\right\}+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0$$

Since $\frac{\mathrm{d}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{d}\varsigma}\cdot\frac{\mathrm{d}\varsigma}{\mathrm{dx}}$, it follows that:

$$\begin{aligned}
&\frac{d}{d\varsigma}\left\{\frac{\left(\varsigma^2-1\right)}{l}\frac{dM_y}{d\varsigma}\right\}\frac{d\varsigma}{dx}+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0\\
&\frac{d}{d\varsigma}\left\{\frac{\left(\varsigma^2-1\right)}{l}\frac{dM_y}{d\varsigma}\right\}\left(-\frac{1}{l}\sin\frac{x}{l}\right)+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0\\
&\frac{d}{d\varsigma}\left\{\frac{-\left(\varsigma^2-1\right)}{l}\frac{dM_y}{d\varsigma}\right\}\frac{1}{l}\sin\frac{x}{l}+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0
\end{aligned} \tag{2.107}$$

Now, since l is a constant, equation (2.107) can be written as:

$$\frac{d}{d\varsigma}\left\{\left(1-\varsigma^2\right)\frac{dM_y}{d\varsigma}\right\}\frac{1}{l^2}\sin\frac{x}{l}+\frac{1}{l^2}\sin\frac{x}{l}n(n+1)M_y=0 \qquad (2.108)$$

Dividing all through by $\frac{1}{l^2}\sin\frac{x}{l}$, it follows that:

$$\frac{d}{d\varsigma}\left\{\left(1-\varsigma^2\right)\frac{dM_y}{d\varsigma}\right\}+n(n+1)M_y=0 \qquad (2.109)$$

$$\left(1-\varsigma^2\right)\frac{d^2M_y}{d\varsigma^2}-2\varsigma\frac{dM_y}{d\varsigma}+n(n+1)M_y=0 \qquad (2.110)$$

This is the Legendre differential equation and has a solution of the form:

$$M_y=C_1P_n(\varsigma)+C_2Q_n(\varsigma) \qquad (2.111)$$

where $P_n(\varsigma)$ are the Legendre polynomials of the first kind (which are regular at finite points) while $Q_n(\varsigma)$ are the Legendre Polynomials [56-57] of the second kind (which are singular at ± 1). C_1 and C_2 are constants.

It is worthy of note that $P_n(\varsigma)$ and $Q_n(\varsigma)$ are two linearly independent solutions to the equation (2.110). We can write that:

$$M_y=P_n(\varsigma)=M_{yn}$$

$$\left\{P_n(\varsigma)=P_0\left(\cos\frac{x}{l}\right)\right\}$$

$$P_n(\varsigma)=\sum_{p=0}^{m}\frac{(-1)^p(2n-2p)!}{2^n p!(n-p)!(n-2p)!}\varsigma^{n-2p} \qquad (2.112a)$$

$$M_{yn}(\varsigma)=\sum_{p=0}^{m}\frac{(-1)^{p}(2n-2p)!}{2^{n}\,p!(n-p)!(n-2p)!}\varsigma^{n-2p} \tag{2.112b}$$

$$m=\frac{n}{2}\,or\,\frac{(n-1)}{2}$$

whichever m is an integer. We have noted earlier that this expression implies

$$m=\frac{2n+\left((-1)^{n}-1\right)}{4}$$

Then,

$$M_{yn}(\varsigma)=\sum_{p=0}^{\frac{2n+\left((-1)^{n}-1\right)}{4}}\frac{(-1)^{p}(2n-2p)!}{2^{n}\,p!(n-p)!(n-2p)!}\varsigma^{n-2p}$$

This solution can be written as:

$$M_{yn}(\varsigma)=\frac{(2n-1)(2n-3)\ldots1}{n!}\left\{\varsigma^{n}-\frac{n(n-1)}{2(2n-1)}\varsigma^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot4(2n-1)(2n-3)}\varsigma^{n-4}+\ldots\right\} \tag{2.113}$$

However, for the purpose of the Boubaker Polynomial problem [44, 59], we shall write that:

$$M_{yn}(\varsigma)=\frac{(2n)!}{2^{n}n!n!}\left\{\varsigma^{n}-\frac{n(n-1)}{1!\cdot2(2n-1)}\varsigma^{n-2}+\frac{n(n-1)(n-1)(n-2)(n-3)}{2!\cdot2(2n-1)(2n-2)(2n-3)}\varsigma^{n-4}+\ldots\right\}$$

(The expression of equation (2.113) is based on some sort of definition which is a choice made in order that $P_n(1)=1$).

$$M_{yn}(\varsigma)=\frac{(2n)!}{2^n n!n!}\left\{\varsigma^n-\frac{n(n-1)}{2(2n-1)\cdot(1!)}\varsigma^{n-2}+\frac{n(n-1)(n-1)(n-2)(n-3)}{2(2n-1)(2n-2)(2n-3)\cdot(2!)}\varsigma^{n-4}+\ldots\right\} \quad (2.114)$$

For

$$\frac{(2n)!}{2^n n!n!}=1 \quad (2.115a)$$

Then, $n=0,1$.

This condition will cause all other co-efficient of ς to be equal to zero, so that:

$$M_{yn}(\varsigma)=\varsigma^n \text{ (for } n=0,1) \quad (2.115b)$$

However in addition to condition (2.115a), if we can establish that:

$$\left.\begin{aligned} n-4 &\equiv \frac{n\cdot(n-1)}{2\cdot(2n-1)} \\ n-8 &\equiv \frac{n\cdot(n-1)(n-1)(n-2)}{2\cdot(2n-1)(2n-2)(2n-3)} \\ n-12 &\equiv \frac{n\cdot(n-1)(n-1)(n-2)(n-2)(n-3)}{2\cdot(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)} \end{aligned}\right\} \quad (2.116)$$

It follows that:

$$M_{yn}(\varsigma)=1\cdot\varsigma^n-(n-4)\varsigma^{n-2}+\frac{(n-8)(n-3)}{2!}\varsigma^{n-4}-\frac{(n-12)(n-4)(n-5)}{3!}\varsigma^{n-6}+\ldots$$

Since the sum:

$$\frac{(n-8)(n-3)}{2!}\varsigma^{n-4}-\frac{(n-12)(n-4)(n-5)}{3!}\varsigma^{n-6}+\ldots$$

is given by

$$\sum_{p=2}^{\frac{2n+\left((-1)^n-1\right)}{4}}\left\{\frac{(n-4p)}{p!}\prod_{j=p+1}^{2p-1}(n-j)\right\}(-1)^p\varsigma^{n-2p}$$

we obtain:

$$\begin{aligned}M_{yn}(\varsigma)&=1\cdot\varsigma^{n}-(n-4)\varsigma^{n-2}\\&+\sum_{p=2}^{\frac{2n+\left((-1)^n-1\right)}{4}}\left\{\frac{(n-4p)}{p!}\prod_{j=p+1}^{2p-1}(n-j)\right\}(-1)^p\varsigma^{n-2p}\end{aligned} \tag{2.117}$$

If the assumption would hold for the Legendre polynomials of the second kind, the procedure can be extended to the transverse magnetization in the form:

$$M_{yn}(\varsigma)=Q_n(\varsigma) \tag{2.118}$$

2.12 Sturm - Liouville Problem

The Legendre polynomials have the orthogonality property expressed as follows:

$$\int_{-1}^{1}P_m(x)P_n(x)dx=\frac{2}{2n+1}\delta_{mn} \tag{2.119}$$

The reason for this orthogonality property is that Legendre differential equation can be viewed as a Sturm-Liouville problem:

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{d}{dx}y(x)\right\}=-\lambda y(x) \tag{2.120}$$

where $y(x)\equiv M_y$ and $\lambda\equiv n(n+1)$.

We shall make an assumption of the form:

$$\frac{T_0}{v} = \frac{1}{l}\cot\frac{x}{l} \tag{2.121}$$

(l is a parameter to be determined). Hence, the time-independent equation:

$$\frac{d^2M_y}{dx^2} + \frac{T_0}{v}\frac{dM_y}{dx} + \frac{S(x)}{v^2}M_y = 0$$

becomes

$$\frac{d^2M_y}{dx^2} + \frac{1}{l}\cot\frac{x}{l}\frac{dM_y}{dx} + \frac{S(x)}{v^2}M_y = 0 \tag{2.122}$$

$$\begin{gathered}
\frac{d^2M_y}{dx^2} + \frac{1}{l}\frac{\cos\frac{x}{l}}{\sin\frac{x}{l}}\frac{dM_y}{dx} + \frac{S(x)}{v^2}M_y = 0 \\
\sin\frac{x}{l}\frac{d^2M_y}{dx^2} + \frac{1}{l}\cos\frac{x}{l}\frac{dM_y}{dx} + \sin\frac{x}{l}\frac{S(x)}{v^2}M_y = 0 \\
\frac{d}{dx}\left(\sin\frac{x}{l}\frac{dM_y}{dx}\right) + \sin\frac{x}{l}\frac{S(x)}{v^2}M_y = 0
\end{gathered} \tag{2.123}$$

If $n(n+1) = l^2 \sin\frac{x}{l}\frac{S(x)}{v^2}$:

$$\sin\frac{x}{l}\frac{S(x)}{v^2} = \frac{1}{l^2}n(n+1)$$

It follows that:

$$\frac{d}{dx}\left(\sin\frac{x}{l}\frac{dM_y}{dx}\right) + \frac{1}{l^2}n(n+1)M_y = 0$$

$$\frac{d}{dx}\left(\sin\frac{x}{l}\frac{dM_y}{dx}\right) = -\frac{1}{l^2}n(n+1)M_y$$

$$l^2 \frac{d}{dx}\left(\sin\frac{x}{l}\frac{dM_y}{dx} \right) = -n(n+1)M_y$$

Since l^2 is not dependent on x, we have:

$$\frac{d}{dx}\left(l^2 \sin\frac{x}{l}\frac{dM_y}{dx} \right) = -n(n+1)M_y \qquad (2.124)$$

This is not exactly the same as equation (2.120), but equation (2.124) can be compared to the form that we stated earlier, that is,

$$\frac{d}{dx}\left\{ p(x)\frac{dy}{dx} \right\} + [q(x) + \lambda r(x)]y = 0$$

Where

$$y(x) \equiv M_y,$$

$$p(x) = l^2 \sin\frac{x}{l}$$

and

$$\left[q(x) + \lambda r(x)\right] = n \cdot (n+1) = l^2 \sin\frac{x}{l}\frac{S(x)}{v^2} = \frac{l^2}{v^2}\sin\frac{x}{l}\left(\gamma^2 B_1^2(x) + T_g\right)$$

However, earlier on, equation (2.109) is given as:

$$\begin{aligned} &\frac{d}{d\varsigma}\left\{ \left(1-\varsigma^2\right)\frac{dM_y}{d\varsigma} \right\} + n(n+1)M_y = 0 \\ &\frac{d}{d\varsigma}\left\{ \left(1-\varsigma^2\right)\frac{dM_y}{d\varsigma} \right\} = -n(n+1)M_y = 0 \end{aligned} \qquad (2.125)$$

Hence, for ς -dependence, we can establish Sturm-Liouvile problem in the form of equation (2.125).

2.13 The Diffusion - Advection Equation

The diffusion - advection equation is a differential equation describing the process of diffusion and advection. For the investigation of the diffusion process of magnetization in a fluid moving at a uniform velocity which is constant in time, we have to take the process of advection into consideration. The equation which describes such a process is known as the Advection equation. The advection equation is the partial differential equation that governs the motion of a conserved scalar as it is advected by a known velocity field. It is derived using the scalar's conservation law, together with Gauss's theorem, and taking the infinitesimal limit.

The diffusion - advection equation (a differential equation describing the process of diffusion and advection) is obtained by adding the advection operator to the main diffusion equation. In the Cartesian coordinates, the advection operator [58] is:

$$\vec{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

where the velocity vector v has components v_χ, v_y and v_z in the x, y and z directions respectively.

Therefore, from Equation (2.18b),

$$\begin{aligned} & v^2 \frac{\partial^2 M_y}{\partial x^2} + 2v \frac{\partial^2 M_y}{\partial x \partial t} + vT_o \frac{\partial M_y}{\partial x} + T_o \frac{\partial M_y}{\partial t} \\ & + \frac{\partial^2 M_y}{\partial t^2} + \left\{ T_g + \gamma^2 B_1^{\ 2}(x,t) \right\} M_y \\ & = F_o \gamma B_1(x,t) \end{aligned}$$

we can write:

$$\frac{\partial^2 M_y}{\partial t^2} + 2v\frac{\partial^2 M_y}{\partial x \partial t} + \left\{T_g + \gamma^2 B_1^{\ 2}(x,t)\right\}M_y = 0 \tag{2.126}$$

It then follows that:

$$v^2 \frac{\partial^2 M_y}{\partial x^2} + vT_o \frac{\partial M_y}{\partial x} + T_o \frac{\partial M_y}{\partial t} = F_o \gamma B_1(x,t) \tag{2.127}$$

$$vT_o \frac{\partial M_y}{\partial x} + T_o \frac{\partial M_y}{\partial t} = -v^2 \frac{\partial^2 M_y}{\partial x^2} + F_o \gamma B_1(x,t) \tag{2.128}$$

If we multiply Equation (2.128) all through by $\frac{1}{T_0}$, it follows therefore that:

$$v\frac{\partial M_y}{\partial x} + \frac{\partial M_y}{\partial t} = -\frac{v^2}{T_o}\frac{\partial^2 M_y}{\partial x^2} + \frac{F_o}{T_o}\gamma B_1(x,t) \tag{2.129}$$

where

$$D = -\frac{v^2}{T_o} \tag{2.130}$$

hence,

$$v\frac{\partial M_y}{\partial x} + \frac{\partial M_y}{\partial t} = D\frac{\partial^2 M_y}{\partial x^2} + \frac{F_o}{T_o}\gamma B_1(x,t) \tag{2.131}$$

Provided that D is the diffusion coefficient, and since v is the fluid velocity, equation (2.131) is the diffusion – advection equation for the NMR transverse magnetization. It is very interesting to note that equation (2.131) exactly match the advection equation without any special transformation whatsoever.

2.14 The Euler NMR Equation

Based on equations (2.18b) and (2.29), we can define for constant fluid velocity v, and $\gamma^2 B_1^2$ field:

$$\eta T_o = -T_g; \qquad \nu\mu T_o = \gamma^2 B_1^2 \tag{2.132}$$

or

$$\eta T_o = \gamma^2 B_1^2; \qquad \nu\mu T_o = -T_g \tag{2.133}$$

When the maximum NMR signal is received at maximum rf B_1 field and $M_o = 0$, equation (2.18b) becomes:

$$v^2 \frac{\partial^2 M_y}{\partial x^2} + 2v \frac{\partial^2 M_y}{\partial x \partial t} + \frac{\partial^2 M_y}{\partial t^2} = 0 \tag{2.134}$$

This equation can also be derived for the following rf limits.

For $\gamma^2 B_1^2 \ll T_g$, equation (2.18b) becomes:

$$v^2 \frac{\partial^2 M_y}{\partial x^2} + 2v \frac{\partial^2 M_y}{\partial x \partial t} + \frac{\partial^2 M_y}{\partial t^2} = 0 \tag{2.135}$$

provided that:

$$\eta T_o + \nu\mu T_o = -T_g \tag{2.136}$$

$$\eta = \nu\mu \tag{2.137}$$

$$2\eta T_o = -T_g \tag{2.138}$$

Equation (2.135) is called the Euler's equation.

2.15 Analytical Solutions to the Generalized Bloch NMR Flow Equation

It may be very important to solve equation (2.18b) analytically for various applications as highlighted in the editorial introduction.

Equation (2.18b) can be written as:

$$\begin{aligned} & v^2\frac{\partial^2 M_y}{\partial x^2}+2v\frac{\partial^2 M_y}{\partial x\partial t}+\frac{\partial^2 M_y}{\partial t^2}+T_o v\frac{\partial M_y}{\partial x} \\ & +T_o\frac{\partial M_y}{\partial t}\left(\gamma^2 B_1^2(x,t)+T_g\right)M_y \\ & =F_o\gamma B_1(x,t) \end{aligned} \tag{2.139}$$

where

$$T_0=\frac{1}{T_1}+\frac{1}{T_2},\ T_g=\frac{1}{T_1T_2},\ F_0=\frac{M_0}{T_1}$$

Because of the difficulty involved in solving a differential equation which does not have its coefficient to be constant, we shall consider the case where:

$$T_g >> \gamma^2 B_1^2(x,t)$$

Equation (2.139) becomes:

$$\begin{aligned} & v^2\frac{\partial^2 M_y}{\partial x^2}+2v\frac{\partial^2 M_y}{\partial x\partial t}+\frac{\partial^2 M_y}{\partial t^2}+T_0 v\frac{\partial M_y}{\partial x}+T_0\frac{\partial M_y}{\partial t}+T_g M_y \\ & =F_0\gamma B_1(x,t) \end{aligned} \tag{2.140}$$

Assuming a solution of the form:

$$M_y(x,t)=\sum_{n=1}^{\infty}U_n(t)\sin\frac{n\pi x}{\rho}+\sum_{n=1}^{\infty}U_n(t)\cos\frac{n\pi x}{\rho} \tag{2.141}$$

(ρ is a term which has the same dimension as x). It then follows that:

$$\frac{\partial M_y}{\partial x} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)U_n(t)\cos\frac{n\pi x}{\rho} - \sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)U_n(t)\sin\frac{n\pi x}{\rho}$$

$$\frac{\partial^2 M_y}{\partial x} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)^2 U_n(t)\sin\frac{n\pi x}{\rho} - \sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)^2 U_n(t)\cos\frac{n\pi x}{\rho}$$

$$\frac{\partial^2 M_y}{\partial x\partial t} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}\cos\frac{n\pi x}{\rho} - \sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}\sin\frac{n\pi x}{\rho}$$

$$\frac{\partial M_y}{\partial t} = \sum_{n=1}^{\infty}\frac{dU_n}{dt}\sin\frac{n\pi x}{\rho} + \sum_{n=1}^{\infty}\frac{dU_n}{dt}\cos\frac{n\pi x}{\rho}$$

$$\frac{\partial^2 M_y}{\partial t^2} = \sum_{n=1}^{\infty}\frac{d^2U_n}{dt^2}\sin\frac{n\pi x}{\rho} + \sum_{n=1}^{\infty}\frac{d^2U_n}{dt^2}\cos\frac{n\pi x}{\rho}$$

Equation (2.140) then becomes:

$$\begin{aligned}
&-v^2\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)^2 U_n\sin\frac{n\pi x}{\rho} - v^2\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)^2 U_n \\
&\cos\frac{n\pi x}{\rho} + 2v\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}\cos\frac{n\pi x}{\rho} \\
&-2v\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}\sin\frac{n\pi x}{\rho} + \sum_{n=1}^{\infty}\frac{d^2U_n}{dt^2}\sin\frac{n\pi x}{\rho} + \sum_{n=1}^{\infty}\frac{d^2U_n}{dt^2}\cos\frac{n\pi x}{\rho} \\
&+T_0v\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)U_n\cos\frac{n\pi x}{\rho} - T_0v\sum_{n=1}^{\infty}\left(\frac{n\pi}{\rho}\right)U_n \\
&\sin\frac{n\pi x}{\rho} + T_0\sum_{n=1}^{\infty}\frac{dU_n}{dt}\sin\frac{n\pi x}{\rho} + T_0\sum_{n=1}^{\infty}\frac{dU_n}{dt}\cos\frac{n\pi x}{\rho} + T_g\sum_{n=1}^{\infty}U_n \\
&\sin\frac{n\pi x}{\rho} + T_g\sum_{n-=1}^{\infty}U_n\cos\frac{n\pi x}{\rho} = F_o\gamma B_1(x,t)
\end{aligned} \tag{2.142}$$

$$\sum_{N=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}\sin\frac{n\pi x}{\rho}+T_0\frac{dU_n}{dt}\sin\frac{n\pi x}{\rho}-2v\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}\sin\frac{n\pi x}{\rho}\right.$$
$$+T_gU_n\sin\frac{n\pi x}{\rho}-T_0v\left(\frac{n\pi}{\rho}\right)U_n\sin\frac{n\pi x}{\rho}-v^2\left(\frac{n\pi}{\rho}\right)^2U_n\sin\frac{n\pi x}{\rho}$$
$$+T_0\frac{dU_n}{dt}\cos\frac{n\pi x}{\rho}+2v\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}\cos\frac{n\pi x}{\rho}+T_gU_n \qquad (2.143)$$
$$\cos\frac{n\lambda x}{\rho}+T_0v\left(\frac{n\lambda}{\rho}\right)U_n\cos\frac{n\pi x}{\rho}-v^2\left(\frac{n\pi}{\rho}\right)^2U_n$$
$$\left.\cos\frac{n\lambda x}{\rho}+T_0v\left(\frac{n\lambda}{\rho}\right)U_n\cos\frac{n\pi x}{\rho}-v^2\left(\frac{n\pi}{\rho}\right)^2U_n\cos\frac{n\pi x}{\rho}\right\}$$

$$\sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+T_0\frac{dU_n}{dt}+2v\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}+T_gU_n-T_0v\left(\frac{n\pi}{\rho}\right)U_n-v^2\left(\frac{n\pi}{\rho}\right)^2U_n\right\}\sin\frac{n\pi x}{\rho}$$
$$+\sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+T_0\frac{dU_n}{dt}+2v\left(\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}+T_gU_n+T_0v\left(\frac{n\pi}{\rho}\right)U_n-v^2\left(\frac{n\pi}{\rho}\right)^2U_n\right\}\cos\frac{n\pi x}{\rho}$$
$$=F_0\,\gamma B_1(x,t)$$
$$(2.144)$$
$$\sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+(T_0-2v\frac{n\pi}{\rho})\frac{dU_n}{dt}+(T_g-T_0v\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2)U_n\right\}\sin\frac{n\pi x}{\rho}$$
$$\sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+(T_0+2v\frac{n\pi}{\rho})\frac{dU_n}{dt}+(T_g+T_0v\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2)U_n\right\}\cos\frac{n\pi x}{\rho}$$
$$=F_o\gamma B_1(x,t)$$

Multiplying equation: (2.144) all through by $\cos\frac{p\pi x}{\rho}$, it follows that:

$$\sum_{n=1}^{\alpha}\left\{\frac{d^2U_n}{dt^2}+\left(T_0-2v\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g-T_0v\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2U_n\right\}$$
$$\sin\frac{n\pi x}{\rho}\cos\frac{p\pi x}{\rho}+$$
$$\sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+\left(T_0+2\frac{vn\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g+T_ov\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2\right)U_n\right\} \tag{2.145}$$
$$\cos\frac{n\pi x}{\rho}\cos\frac{p\pi x}{\rho}=F_0\gamma B_1(x,t)\cos\frac{p\pi x}{\rho}$$

Integrating both sides from 0 to ρ with respect to x, we have:

$$\int_0^{\rho}\sum_{n=1}^{\alpha}\left\{\frac{d^2U_n}{dt^2}+\left(T_0-2v\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g-T_0v\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2U_n\right\}$$
$$\sin\frac{n\pi x}{\rho}\cos\frac{p\pi x}{\rho}dx+$$
$$\int_0^{\rho}\sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+\left(T_0+2\frac{vn\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g+T_ov\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2\right)U_n\right\} \tag{2.146}$$
$$\cos\frac{n\pi x}{\rho}\cos\frac{p\pi x}{\rho}dx$$
$$=\int_0^{\rho}F_0\gamma B_1(x,t)\cos\frac{p\pi x}{\rho}dx$$

However, for the integral on the LHS of equation (2.146) to be valid, p = n, so that:

$$\int_0^\rho \sum_{n=1}^{\alpha}\left\{\frac{d^2U_n}{dt^2}+\left(T_0-2v\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g-T_0v\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2 U_n\right\}$$
$$\sin\frac{n\pi x}{\rho}\cos\frac{n\pi x}{\rho}dx+$$
$$\int_0^\rho \sum_{n=1}^{\infty}\left\{\frac{d^2U_n}{dt^2}+\left(T_0+2\frac{vn\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g+T_ov\left(\frac{n\pi}{\rho}\right)-v^2\left(\frac{n\pi}{\rho}\right)^2\right)U_n\right\} \tag{2.147}$$
$$\cos^2\frac{n\pi x}{\rho}dx$$
$$=\int_0^\rho F_0\gamma B_1(x,t)\cos\frac{n\pi x}{\rho}dx$$

$$\frac{d^2U_n}{dt^2}+\left(T_0-2v\frac{n\pi}{p}\right)\frac{dU_n}{dt}+\left(T_g-T_0v\frac{n\pi}{\rho}-v^2\left(\frac{n\pi}{\rho}\right)^2\right)U_n$$
$$\int_0^\rho \sin\frac{n\pi x}{\rho}\cos\frac{n\pi x}{\rho}dx$$
$$+\frac{d^2U_n}{dt^2}+\left(T_0+2v\frac{n\pi}{\rho}\right)\frac{dU_n}{dt}+\left(T_g+T_0v\frac{n\pi}{\rho}-v^2\left(\frac{n\pi}{\rho}\right)^2\right)U_n \tag{2.148}$$
$$\int_o^\rho \cos^2\frac{n\pi x}{\rho}dx$$
$$=\int_0^\rho F_o\gamma B_1(x,t)\ \cos\frac{n\pi x}{\rho}dx$$

but

$$\int_0^\rho \sin\frac{n\pi x}{\rho}\cos\frac{n\pi x}{\rho}dx$$
$$=\int_0^\rho \frac{1}{2}\sin\frac{2n\pi x}{\rho}dx=\frac{1}{2}\frac{\rho}{2n\pi}\left[-\cos 2\frac{n\pi x}{\rho}\right]_o^\rho$$
$$=\frac{\rho}{4n\pi}\left[-\cos 2n\pi+1\right]=0$$

and

$$\int_0^{\rho} \cos^2 \frac{n\pi x}{\rho} dx$$
$$= \int_0^{\rho} \frac{1}{2}\left(1 + \cos\frac{2n\pi x}{\rho}\right) dx = \frac{1}{2}\left[x + \frac{\rho}{2n\pi}\sin\frac{2n\pi x}{\rho}\right]_o^{\rho}$$
$$= \frac{1}{2}\left[\rho + \frac{\rho}{2n\pi}\sin 2n\pi - 0\right] = \frac{\rho}{2}$$

Equation (2.148) becomes:

$$\frac{\rho}{2}\frac{d^2U_n}{dt^2} + \frac{\rho}{2}\left(T_0 + 2v\frac{n\pi}{\rho}\right)\frac{dU_n}{dt} + \frac{\rho}{2}\left[T_g + T_o v\frac{n\pi}{\rho} - v^2\left(\frac{n\pi}{\rho}\right)^2\right]U_n$$
$$= F_0 \int_0^{\rho} \gamma B_1(x,t)\cos\frac{n\pi x}{\rho} dx \tag{2.149}$$
$$\frac{d^2U_n}{dt^2} + \left(T_o + 2v\frac{n\pi}{\rho}\right)\frac{dU_n}{dt} + \left(T_o + T_o v\frac{n\pi}{\rho} - v^2\left(\frac{n\pi}{\rho}\right)^2\right)U_n$$
$$= \frac{2F_o}{\rho}\int_o^{\rho} \gamma B_1(x,t)\cos\frac{n\pi x}{\rho} dx$$

We can write:

$$\frac{d^2U_n}{dt^2} + 2\tau\frac{dU_n}{dt} + \xi^2 U_n = \frac{2F_o}{\rho}\int_o^{\rho} \gamma B_1(x,t)\cos\frac{n\pi x}{\rho} dx \tag{2.150}$$

where $2\tau = T_0 + 2v\frac{n\pi}{\rho}$, $\xi^2 = T_g + T_o v\frac{n\pi}{\rho} - v^2\left(\frac{n\pi}{\rho}\right)^2$.

The homogeneous equation of equation (2.150) is given as:

$$\frac{d^2U_n}{dt^2} + 2\tau\frac{dU_n}{dt} + \xi^2 U_n = 0 \tag{2.151}$$

Assuming a solution of the form:

$$U_n(t) = Ae^{\beta t}$$

$$\frac{dU_n}{dt} = \beta A e^{\beta t}$$

$$\frac{d^2 U_n}{dt^2} = \beta^2 A e^{\beta t}$$

Equation (2.151) becomes:

$$\beta^2 A e^{\beta t} + 2y\left(\beta A e^{\beta t}\right) + \xi^2 A e^{\beta t} = 0$$

$$\beta^2 + 2\tau\beta + \xi^2 = 0$$

It follows that:

$$\beta = -2\tau \pm \left(\frac{4\tau^2 - 4\xi^2}{2}\right)^{1/2}$$

$$\beta = -2\tau \pm 2\left(\frac{\tau^2 - \xi^2}{2}\right)^{1/2}$$

if we let $\tau^2 - \xi^2 = \Delta^2, \beta = -2\tau \pm \frac{2\Delta}{2}$

$$\beta = -\tau + \Delta \text{ or } \beta = -\tau + -\Delta$$

Therefore,

$$\begin{aligned} U_n(t) &= A_1 e^{(-\tau+\Delta)t} + A_2 e^{(-\tau-\Delta)t} \\ &= A_1 e^{-\tau t} e^{\Delta t} + A_2 e^{-\tau t} e^{-\Delta t} \\ &= e^{-\tau t}\left[A_1 e^{\Delta t} + A_2 e^{-\Delta t}\right] \\ U_n(t) &= (a \cosh \Delta_n t + b \sinh \Delta_n t) e^{-\tau t} \end{aligned} \quad (2.152)$$

where $a = A_1 + A_2$, $b = A_1 - A_2$, $e^{\Delta t} = \cosh \Delta t + \sinh \Delta t$, $e^{-\Delta t} = \cosh \Delta t - \sinh \Delta t$.

It is worthy of note that the form of our solution is as a result of the fact that:

$$(2\tau)^2 - 4\xi^2 > 0.$$

Equation (2.152) is the complementary function of equation (2.150).

We shall find the integral solution to the equation (2.150):

$$\frac{d^2U_n}{dt^2} + 2\tau\frac{dU_n}{dt} + \xi^2U_n = \frac{2F_0}{\rho}\int_0^\rho \gamma B_1(x,t)\cos\frac{n\pi x}{\rho}dx \tag{2.153}$$

If we take the Radio frequency field to be a sinusoidal wave travelling towards the right (in the positive x-direction), then we can either have:

Case (i)

$$\gamma B_1(x,t) = A\cos(kx - \omega t) = A\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

Case (ii)

$$\gamma B_1(x,t) = A\sin(kx - \omega t) = A\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

(where A is the amplitude of the rf wave).

When the rf field is just removed, we have:

$$\frac{d^2U_n}{dt^2} + 2\tau\frac{dU_n}{dt} + \xi^2U_n = 0$$

and the solution would essentially be equation (2.152):

$$U_n(t) = [a\cosh\Delta_n t + b\sinh\Delta_n t]e^{-\tau t}$$

It would be recalled:

$$M_y(x,t)=\sum_{n=1}^{\infty}U_n(t)\sin\frac{n\pi x}{\rho}+\sum_{n=1}^{\infty}U_n(t)\cos\frac{n\pi x}{\rho}$$

Applying the initial condition:

$$M_y(x,t)=0;$$

$$M_y(x,0)=\sum_{n=1}^{\infty}U_n(0)\sin\frac{n\pi x}{\rho}+\sum_{n=1}^{\infty}U_n(0)\cos\frac{n\pi x}{\rho}=0$$

since $\sin\frac{n\pi x}{\rho}\neq 0$ and $\cos\frac{n\pi x}{\rho}\neq 0$

$$U_n(0)=0$$

$$[a\cosh 0+b\sinh 0]e^{0}=0$$

$$a\cosh 0+b\sinh 0=0$$

$$a=0$$

(since $\cosh 0\neq 0$)

Therefore,

$$U_n(t)=(b\sinh\Delta_n t)e^{-\tau t}$$
$$U_n(t)=be^{-\tau t}\sinh\Delta_n t \tag{2.154}$$

then

$$M_y(x,t)=\sum_{n=1}^{\infty}be^{-\tau t}\sinh\Delta_n t\sin\frac{n\pi x}{\rho}+\sum_{n=1}^{\infty}be^{-\tau t}\sinh\Delta_n t\cos\frac{n\pi x}{\rho} \tag{2.155}$$

(where b is an arbitrary constant).

It is important to note that this method of solution to the problem requires that ρ be a sort of boundary condition parameter such that ρ tells us either:

(a) how far the radio frequency wave can travel in the x-direction, or

(b) how far we can allow the radio frequency wave to travel in the x-direction.

The value of ρ can then be determined by the theory of NMR Physics or in an NMR experiment.

Secondly, this method of solution will work best if $\rho = m\lambda$ (where λ is the wavelength of the radio frequency wave and m is a number whose value must be determined).

We shall evaluate the integral in equation (2.153) in order to determine the particular solution and hence the general solution of $U_n(t)$ when $B_1(x,t) \neq 0$. The integral cannot be solved unless m is known. However, if we assume that m=1, so that we will have a representation of what the actual solution (for which the value of m is fixed) looks like:

Case (i)

$$\gamma B_1(x,t) = A\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

Equation (2.153) becomes:

$$\frac{d^2U_n}{dt^2} + 2\tau\frac{dU_n}{dt} + \xi^2 U_n = \frac{2F_o}{\lambda} A\int_0^\rho \cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}dx. \qquad (2.156)$$

In the integral,

$$\frac{2AF_0}{\lambda}\int_o^\lambda \cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}dx \qquad (2.157)$$

we let:

$$I_1 = \int_o^{\lambda} \cos\left(\frac{2\pi x}{\lambda} - \omega t\right) \cos\frac{n\pi x}{\lambda} dx$$

$$= \left[\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\frac{\lambda}{n\pi}\sin\frac{n\pi x}{\lambda}\right]_0^{\lambda} - \int_0^{\lambda}\left[-\frac{2\pi}{\lambda}\sin\left(\frac{2\pi x}{\lambda} - wt\right)\frac{\lambda}{n\pi}\sin\frac{n\pi x}{\lambda}\right]dx$$

$$= \frac{\lambda}{n\pi}\cos\left(2\pi - \omega t\right)\sin n\pi - 0 + \frac{2\pi}{\lambda}\frac{\lambda}{n\pi}\int_0^{\lambda}\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\sin\frac{n\pi x}{\lambda}dx$$

but Sin $n\pi = 0$:

$$I_1 = \frac{2}{n}\int_0^{\lambda}\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\sin\frac{n\pi x}{\lambda}dx$$

$$= \frac{2}{n}\left\{\left[\text{Sin}\left(\frac{2\pi x}{\lambda} - \omega t\right).-\frac{\lambda}{n\pi}\cos\frac{n\pi x}{\lambda}\right]_0^{\lambda} - \int_0^{\lambda}\left[\frac{2\pi x}{\lambda}\cos\left(\frac{2\pi x}{\lambda} - \omega t\right).-\frac{\lambda}{n\pi}\cos\frac{n\pi x}{\lambda}dx\right]\right\}$$

$$= \frac{2}{n}\left\{\left[-\frac{\lambda}{n\pi}\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}\right]_0^{\lambda} + \frac{2}{n}\int_0^{\lambda}\left[\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}dx\right]\right\}$$

$$I_1 = \frac{2}{n}\left\{-\frac{\lambda}{n\pi}\sin\left(2\pi - \omega t\right) + \frac{\lambda}{n\pi}\sin\left(-\omega t\right)\cos 0 + \frac{2}{n}I_1\right\}$$

$$I_1 = \frac{2\lambda}{n^2\pi}\sin\left(2\pi - \omega t\right)\cos n\pi + \frac{2\lambda}{n^2\pi}\sin\left(-\omega t\right) + \frac{4}{n^2}I_1$$

$$I_1 = -\frac{2\lambda}{n^2\pi}\sin\left(2\pi - \omega t\right)\cos n\pi - \frac{2\lambda}{n^2\pi}\sin\omega t + \frac{4}{n^2}I_1$$

but since $\sin(-\omega t) = -\sin\omega t$ and,
$\sin(2\pi - \omega t) = \sin 2\pi\cos\omega t - \cos 2\pi\sin\omega t = -\sin\omega t$,

we have:

$$I_1 = -\frac{2\lambda}{\left(n^2 - 4\right)\pi}\left(-\cos n\pi\sin\omega t + \sin\omega t\right)$$

$$I_1 = \frac{2\lambda}{\left(n^2 - 4\right)\pi}\left(\cos n\pi - 1\right)\sin\omega t$$

The integral (equation (2.17)) becomes:

$$\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)\sin\omega t$$

and we write equation (2.156) as:

$$\frac{d^2U_n}{dt^2}+2\tau\frac{dU_n}{dt}+\xi^2 Un=\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)\sin\omega t \tag{2.158}$$

If we assume a solution,

$$U_n\left(t\right)=P\cos\omega t+Q\sin\omega t \quad \text{(Particular solution)}$$

$$\frac{dU_n}{dt}=-\omega P\sin\omega t+\omega Q\cos\omega t$$

$$\frac{d^2U_n}{dt^2}=-\omega^2 P\cos\omega t-\omega^2 Q\sin\omega t$$

Equation (2.158) becomes:

$$\left(-\omega^2 P\cos\omega t-\omega^2 Q\sin\omega t\right)-2\omega\tau P\sin\omega t+2\omega\tau Q\cos\omega t+\xi^2 P\cos\omega t+\xi^2 Q\sin\omega t$$

$$=\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)\sin\omega t\left[\left(\xi^2-\omega^2\right)P+2\omega\tau Q\right]\cos\omega t$$

$$+\left[\left(\xi^2-\omega^2\right)Q-2\omega\tau P\right]\sin\omega t\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)\sin\omega t$$

$$(\xi^2-\omega^2)P+2\omega\tau Q=0$$

$$P=-\frac{2\omega\tau Q}{\left(\xi^2-\omega^2\right)}$$

$$\left(\xi^2-\omega^2\right)P+2\omega\tau P=\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)$$

$$\left(\xi^2-\omega^2\right)Q+\frac{4\omega^2\tau^2 Q}{\left(\xi^2-\omega^2\right)}=\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)$$

$$\frac{\left(\xi^2-\omega^2\right)Q+4\omega^2\tau^2 Q}{(\xi^2-\omega^2)}=\frac{4AF_0}{\left(n^2-4\right)\pi}\left(\cos n\pi-1\right)$$

$$Q=\frac{4AF_0\left(\cos n\pi-1\right)}{\left(n^2-4\right)\pi}\frac{\left(\xi^2-\omega^2\right)}{\left(\xi^2-\omega^2\right)^2+4\omega^2\tau^2}$$

$$P=\frac{-2\omega\tau}{\left(\xi^2-\omega^2\right)}\frac{4AF_o\left(\cos n\pi-1\right)}{\left(n^2-4\right)\pi}\frac{\left(\xi^2-\omega^2\right)}{\left(\xi^2-\omega^2\right)^2+4\omega^2\tau^2}$$
$$=-\frac{4AF_o\left(\cos n\pi-1\right)}{\left(n^2-4\right)\pi}\frac{2\omega\tau}{\left(\xi^2-\omega^2\right)^2+4\omega^2\tau^2}$$

Therefore,

$$\begin{aligned}U_n\left(t\right)=&-\frac{4AF_0\left(\cos n\pi-1\right)}{\left(n^2-4\right)\pi}\left\{\frac{2\omega\tau}{\left(\xi^2-\omega^2\right)^2+4\omega^2\tau^2}\right\}\cos\omega t\\&+\frac{4AF_0\left(\cos n\pi-1\right)}{\left(n^2-4\right)\pi}\left\{\frac{\left(\xi^2-\omega^2\right)}{\left(\xi^2-\omega^2\right)^2+4\omega^2\tau^2}\right\}\sin\omega t\end{aligned}\tag{2.159}$$

Hence the general solution for $U_n(t)$ is:

$$U_n(t) = \left[a\cosh\Delta_n t + b\sinh\Delta_n t\right]e^{-\tau t} - \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}$$
$$\left\{\frac{2\omega\tau}{(\xi^2-\omega^2)^2 + 4\omega^2\tau^2}\right\}\cos\omega t + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi} \tag{2.160}$$
$$\left\{\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2 + 4\omega^2\tau^2}\right\}\sin\omega t$$

Using the initial condition, $M_y(x,0) = 0$, it follows that:

$$M_y(x,0) = \sum_{n=1}^{\infty} U_n(0)\sin\frac{n\pi x}{\lambda} + \sum_{n=1}^{\infty} U_n(0)\cos\frac{n\pi x}{\lambda} = 0$$

where $U_n(0) = 0$. That is,

$$\left[a\cosh 0 + b\sinh 0\right]e^0 - \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\left\{\frac{2\omega\tau}{(\xi^2-\omega^2)^2 + 4\omega^2\tau^2}\right\}\cos 0$$
$$+\frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\left\{\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2 + 4\omega^2\tau^2}\right\}\sin 0 = 0 \ a = 0, \tag{2.161}$$
$$\frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\left\{\frac{2\omega\tau}{(\xi^2-\omega^2)^2 + 4\omega^2\tau^2}\right\} = 0$$

then,

$$U_n(t) = be^{-\tau t}\sinh\Delta_n t + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2 + 4\omega^2\tau^2}\sin\omega t$$

Therefore,

$$
\begin{aligned}
M_y(x,t) &= \sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\sin\frac{n\pi x}{\lambda} \\
&\quad +4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)+4\omega^2\tau^2}\sin\omega t\sin\frac{n\pi x}{\lambda} \\
&\quad +\sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\cos\frac{n\pi x}{\lambda} \\
&\quad +4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)+4\omega^2\tau^2}\sin\omega t\cos\frac{n\pi x}{\lambda}
\end{aligned} \tag{2.162}
$$

$$
\begin{aligned}
M_y(x,t) &= \sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\sin\frac{n\pi x}{\lambda}+\sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\cos\frac{n\pi x}{\lambda} \\
&\quad +4AF_o\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)+4\omega^2\tau^2}\sin\omega t\sin\frac{n\pi x}{\lambda} \\
&\quad +4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)+4\omega^2\tau^2}\sin\omega t\cos\frac{n\pi x}{\lambda}
\end{aligned}
$$

Case (ii)

$$\gamma B_1(x,t) = A\sin\left(\frac{2\pi x}{\lambda}-\omega t\right),$$

Equation (2.153) becomes:

$$\frac{d^2U_n}{dt^2}+2\tau\frac{dU_n}{dt}+\xi^2U_n = \frac{2AF_o}{\lambda}\int_0^{\lambda}\sin\left(\frac{2\pi x}{\lambda}-\omega t\right)\cos\frac{n\pi x}{\lambda}dx \tag{2.163}$$

and

$$\frac{2AF_o}{\lambda}\int_0^{\lambda}\sin\left(\frac{2\pi x}{\lambda}-\omega t\right)\cos\frac{n\pi x}{\lambda}dx \tag{2.164}$$

let

$$I_2 = \int_0^{\lambda} \sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}dx$$

$$I_2 = \left[\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\frac{\lambda}{n\pi}\sin\frac{n\pi x}{\lambda}\right]_0^{\lambda} - \int_0^{\lambda}\frac{2\pi}{\lambda}\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\frac{\lambda}{n\pi}\sin\frac{n\pi x}{\lambda}dx$$

$$= -\frac{2}{n}\int \cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\sin\frac{n\pi x}{\lambda}dx$$

$$I_2 = -\frac{2}{n}\left\{\left[\cos\left(\frac{2\pi x}{\lambda} - \omega t\right).-\frac{\lambda}{n\pi}\cos\frac{n\pi x}{\lambda}\right]_0^{\lambda} - \int_0^{\lambda}\frac{2\pi}{\lambda}\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\frac{\lambda}{n\pi}\cos\frac{n\pi x}{\lambda}dx\right\}$$

$$= \frac{2}{n}\left\{\left[\frac{\lambda}{n\pi}\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}\right]_0^{\lambda} + \frac{2}{n}\int_0^{\lambda}\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}dx\right\}$$

$$= \frac{2}{n}\left\{\left[\frac{\lambda}{n\pi}\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)\cos\frac{n\pi x}{\lambda}\right]_0^{\lambda} + \frac{2}{n}I_2\right\}$$

$$= \frac{2}{n}\left\{\frac{\lambda}{n\pi}\cos(2\pi - \omega t)\cos n\pi - \frac{\lambda}{n\pi}\cos(-\omega t) + \frac{2}{n}I_2\right\}$$

but $\cos(-\omega t) = \cos\omega t$ and $\cos(2\pi - \omega t) = \cos\omega t$, it follows that:

$$I_2 = \frac{2\lambda}{n^2\pi}\cos\omega t\cos n\pi - \frac{2\lambda}{n^2\pi}\cos\omega t + \frac{4}{n^2}I_2$$

Re – arranging the above equation gives:

$$I_2 = \frac{2\lambda}{\left(n^2 - 4\right)\pi}\left(\cos n\pi - 1\right)\cos\omega t$$

Equation (2.164) thus becomes:

$$\frac{4AF_o}{\left(n^2 - 4\right)\pi}\left(\cos n\pi - 1\right)\cos\omega t$$

Then equation (2.153) can be written as:

$$\frac{d^2U_n}{dt^2} + 2\tau\frac{dU_n}{dt} + \xi^2U_n = \frac{4AF_o}{\left(n^2 - 4\right)\pi}\left(\cos n\pi - 1\right)\cos\omega t \tag{2.165}$$

If we assume a solution of the form:

$$U_n(t) = p\cos\omega t + q\sin\omega t$$

$$\frac{dU_n(t)}{dt} = -\omega p\sin\omega t + \omega q\cos\omega t$$

$$\frac{d^2U_n(t)}{dt^2} = -\omega^2 p\cos\omega t - \omega^2 q\sin\omega t$$

Equation (2.165) becomes:

$$\begin{aligned}
&-\omega^2 p\cos\omega t - \omega^2 q\sin\omega t - 2\tau\omega p\sin\omega t + 2\tau\omega q\cos\omega t \\
&+\xi^2 p\cos\omega t + \xi^2 q\sin\omega t \\
&= \frac{4F_0}{\left(n^2-4\right)\pi}\left(\cos n\pi - 1\right)\cos\omega t\left[\left(\xi^2-\omega^2\right)P + 2\tau\omega q\right]\cos\omega t \\
&+\left[\xi^2 - \omega^2 q 2\tau\omega p\right]\sin\omega t \\
&= \frac{4F_0}{\left(n^2-4\right)\pi}\left(\cos n\pi - 1\right)\cos\omega t\left(\xi^2-\omega^2\right)P + 2\tau\omega q \\
&= 0
\end{aligned}$$

$$q = \frac{2\tau\omega}{\left(\xi^2-\omega^2\right)}p$$

$$\left(\xi^2-\omega^2\right)p + 2\tau\omega q = \frac{4AF_o\left(\cos n\pi - 1\right)}{\left(n^2-4\right)\pi}$$

$$\frac{(\xi^2-\omega^2)p + 4\tau^2\omega^2 p}{\left(\xi^2-\omega^2\right)} = \frac{4AF_0\left(\cos n\pi - 1\right)}{\left(n^2-4\right)\pi}$$

$$p = \frac{4AF_0\left(\cos n\pi - 1\right)}{\left(n^2-4\right)\pi}\frac{\left(\xi^2-\omega^2\right)}{(\xi^2-\omega^2)^2 + 4\tau^2\omega^2}$$

$$q = \frac{4AF_0\left(\cos n\pi - 1\right)}{\left(n^2-4\right)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2 + 4\tau^2\omega^2}$$

$$U_n(t) = \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\cos\omega t + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t \quad (2.166)$$

Equation (2.166) is the particular solution. The general solution for $U_n(t)$ is given as:

$$U_n(t) = [a\cosh\Delta_n t + b\sinh\Delta_n t]e^{-\tau t} + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\cos\omega t + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t \quad (2.167)$$

Again, the initial condition, $M_y(x,0) = 0$ requires that:

$$U_n(0) = 0$$

$$[a\cosh 0 + b\sinh 0]e^0 + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\cos 0 + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin 0 = 0$$

This implies:

$$a = 0, \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{(\xi^2-\omega^2)}{(\xi^2-\omega^2)^2+4\tau^2\omega^2} = 0 \quad (2.168)$$

$$U_n(t) = be^{-\tau t}\sinh\Delta_n t + \frac{4AF_0(\cos n\pi - 1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t \quad (2.169)$$

then,

$$M_y(x,t)=\sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\sin\frac{n\pi x}{\lambda}+4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}$$
$$\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t\sin\frac{n\pi x}{\lambda}\sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\cos\frac{n\pi x}{\lambda}$$
$$+4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t\cos\frac{n\pi x}{\lambda}$$
$$M_y(x,t)=\sum_{n=1}^{\infty} be^{-\tau t}\sinh\Delta_n t\sin\frac{n\pi x}{\lambda}+\sum_{n=1}^{\infty} be^{-\tau t}\,\mathrm{Sinh}\Delta_n t\cos\frac{n\pi x}{\lambda}$$
$$+4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t\sin\frac{n\pi x}{\lambda}$$
$$+4AF_0\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{(n^2-4)\pi}\frac{2\omega\tau}{(\xi^2-\omega^2)^2+4\omega^2\tau^2}\sin\omega t\cos\frac{n\pi x}{\lambda} \tag{2.170}$$

In equation (2.170) the following parameters are defined:

(i) $$2\tau=T_0+\frac{2vn\pi}{\lambda}=\frac{T_0}{2}+\frac{vn\pi}{\lambda}$$

$$\tau^2=\frac{T_0}{4}+T_0\frac{vn\pi}{\lambda}+v^2\left(\frac{n\pi}{\lambda}\right)^2$$

(ii) $$\xi^2=T_g+\frac{T_0 vn\pi}{\lambda}-v^2\left(\frac{n\pi}{\lambda}\right)^2$$

(iii)
$$\Delta^2=\tau^2-\xi^2=\frac{T_0{}^2}{4}+v^2\left(\frac{n\pi}{\pi}\right)^2-T_g+v^2\left(\frac{n\pi}{\lambda}\right)^2$$
$$=\frac{T_0{}^2}{4}-T_g+2v^2\left(\frac{n\pi}{\lambda}\right)^2$$
$$\Delta=\sqrt{\frac{T_o^2}{4}-T_g+2v^2\left(\frac{n\pi}{\lambda}\right)^2}$$

(iv) $$2\omega\tau = 2\omega\left(\frac{T_o}{2} + \frac{vn\pi}{\lambda}\right) = \omega T_o + \frac{2\omega vn\pi}{\lambda}$$

$$4\omega^2\tau^2 = \left(\omega T_o + \frac{2\omega vn\pi}{\lambda}\right)^2 = \omega^2 T_o^2 + 4\omega^2 T_o \frac{vn\pi}{\lambda} + 4\omega^2 v^2\left(\frac{n\pi}{\lambda}\right)^2$$

2.16 Solutions to the NMR Travellling Wave Equation

Based on equation (2.35), the NMR wave equation can be written in the form:

$$v^2\frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_y}{\partial t^2} = F_O\gamma B_1(\mathrm{t}) \qquad (2.171)$$

or

$$v^2\frac{\partial^2 M_y}{\partial x^2} = -\frac{\partial^2 M_y}{\partial t^2} + F_O\gamma B_1(\mathrm{t})$$

$$v^2\frac{\partial^2 M_y}{\partial x^2} = F_O\gamma B_1(\mathrm{t}) - \frac{\partial^2 M_y}{\partial t^2} \qquad (2.172)$$

In generalizing equation (2.172), we write:

$$\mathrm{v}^2\nabla^2 M_y = F_O\gamma B_1(\mathrm{t}) - \frac{\partial^2 M_y}{\partial t^2} \qquad (2.173)$$

In the polar coordinate, equation (2.173) becomes:

$$\mathrm{v}^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}\right) = F_O\gamma B_1(\mathrm{t}) - \frac{\partial^2 M_y}{\partial t^2} \qquad (2.174)$$

Case (1)

For a steady state condition, $\frac{\partial^2 M_y}{\partial t^2} = 0$, and equation (2.174) becomes:

$$\mathrm{v}^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}\right) = F_O\gamma B_1(\mathrm{t})$$

(a) The homogeneous equation gives:

$$\mathrm{v}^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}\right) = 0 \quad [M_0=0 \text{ or } \gamma B_1(t) = 0\,]$$

And the solution follows exactly as in the case of the diffusion equation, that is:

$$M_y(\mathrm{r}, \phi) = \frac{A_0}{2} + \sum_{M=1}^{\infty} r^m \left(A_m \,\mathrm{Cos} m\phi + \mathrm{B_m} \sin \mathrm{m}\phi\right) \tag{2.175}$$

(b) The inhomogeneous equation is:

$$\mathrm{v}^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}\right) = F_O \gamma B_1(\mathrm{t})$$

We need to give a definition to the function $\gamma B_1(\mathrm{t})$ before solving it.

Case (ii)

If $\frac{\partial M_y}{\partial t} \neq 0$ and $m_0 = 0$ or $\gamma B_1(t) = 0$, (2.4) becomes:

$$\mathrm{v}^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}\right) = -\frac{\partial^2 M_y}{\partial t^2} \tag{2.176}$$

We can then write that:

$$\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2} = -\frac{1}{V^2}\frac{\partial^2 M_y}{\partial t^2} \tag{2.177}$$

By the method of separation of variables we have:

$$M_y = \mathrm{R(r)}\,\Phi(\phi) T(t)$$

$$\frac{\partial M_y}{\partial r} = \Phi\mathrm{T}\frac{\partial \mathrm{R}}{\partial \mathrm{r}},\ \frac{\partial^2 M_y}{\partial r^2} = \Phi\mathrm{T}\frac{\partial^2 R}{\partial r^2},\ \frac{\partial M_y}{\partial \phi} = RT\frac{\partial \Phi}{\partial \phi},\ \frac{\partial^2 M_y}{\partial \phi^2} = RT\frac{\partial^2 \Phi}{\partial \phi^2}$$

$$\frac{\partial M_y}{\partial t} = R\Phi\frac{\partial T}{\partial t},\ \frac{\partial^2 M_y}{\partial t^2} = R\Phi\frac{\partial^2 T}{\partial t^2}$$

Equation (2.177) becomes:

$$\Phi T\frac{\partial^2 R}{\partial r^2} + \frac{\Phi \mathrm{T}}{\mathrm{r}}\frac{\partial R}{\partial r} + \frac{RT}{r^2}\frac{\partial^2 \Phi}{\partial \Phi^2} = -\frac{R\Phi}{V^2}\frac{\partial^2 T}{\partial t^2}$$

Multiplying all through by $\frac{1}{R\Phi\mathrm{T}}$,

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} + \frac{1}{r^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{1}{V^2}.\frac{1}{T}\frac{\partial^2 T}{\partial t^2}$$

Both the RHS and LHS must be equal to a constant: $-\alpha^2$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} + \frac{1}{r^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{1}{V^2}.\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\alpha^2$$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} + \frac{1}{r^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -\alpha^2$$

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial \mathrm{R}}{\partial \mathrm{r}} + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -\alpha^2 r^2 \quad (2.178)$$

$$\frac{\partial^2 T}{\partial t^2} = \alpha^2 v^2 T \quad (2.179)$$

If we assume that:

$$\mathrm{T} = Ce^{mt}$$

Then equation (2.179) becomes:

$$m^2 = \alpha^2 v^2$$

$$m = \pm\,\alpha v$$

$$\mathrm{T(t)} = C_1 e^{\alpha v t} + C_2 e^{-\alpha v t} \quad (2.180)$$

Equation (2.178) can be written as:

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial \mathrm{R}}{\partial \mathrm{r}} + \alpha^2 r^2 = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}$$

This equation must also be equal to a constant β^2,

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial \mathrm{R}}{\partial \mathrm{r}} + \alpha^2 r^2 = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = \beta^2$$

Where

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial \mathrm{R}}{\partial \mathrm{r}} + \alpha^2 r^2 = \beta^2 \tag{2.181}$$

and

$$-\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = \beta^2 \tag{2.182}$$

From (2.181), we obtain:

$$r^2\frac{\partial^2 R}{\partial r^2} + r\,\frac{\partial R}{\partial r} + (\,\alpha^2 r^2 - \beta^2)\,\mathrm{R} = 0$$

A solution of which is given as:

$$\mathrm{R(r)} = A_1 J_\beta(\alpha r) + B_1 Y_\beta(\alpha r) \tag{2.183}$$

From (2.182), we write that:

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -\beta^2 \Phi \tag{2.184}$$

If we assume that:

$$\Phi = p\mathrm{e}^{\mathrm{n}\phi},$$

it follows from (2.184) that:

$$\mathrm{n}^2 = -\beta^2$$

$$\mathrm{n} = \pm\, i\beta$$

Therefore, $\Phi(\phi) = \mathrm{P}_1 \mathrm{e}^{i\beta\phi} + \mathrm{P}_2 \mathrm{e}^{-i\beta\phi}$.

From which we have that:

$$\Phi(\phi) = Pcos\beta\phi + \mathrm{Q}\sin\beta\phi \tag{2.185}$$

$$\mathrm{P} = \mathrm{P}_1 + \mathrm{P}_2,\ \mathrm{Q} = i(\mathrm{P}_1 - \mathrm{P}_2)$$

Therefore,

$$\mathrm{M_y} = \{\mathrm{C_1 e^{\alpha vt}} + \mathrm{C_2 e^{-\alpha vt}}\}\,\{\mathrm{A_1 J_\beta(\alpha r)} + \mathrm{B_1 Y_\beta(\alpha r)}\}\,\{\mathrm{Pcos\beta\phi} + \mathrm{Qsin\beta\phi} \tag{2.186}$$

We can then make use of the appropriate boundary condition as required.

Case (iii)

If $\frac{\partial M_y}{\partial t} \neq 0$ and, $M_o \neq 0$ and $\gamma B_1(t) \neq 0$,

$$v^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}\right) = -\partial^2\frac{M_y}{\partial t^2} + F_O\gamma B_1(t) \qquad (2.187)$$

If we write that:

$$M_y(r,\varphi,t) = X(r,\varphi,t) + Y(t)$$

$$\frac{\partial M_y}{\partial r} = \frac{\partial X}{\partial r}, \frac{\partial^2 M_y}{\partial r^2} = \frac{\partial^2 X}{\partial r^2}, \quad \frac{\partial M_y}{\partial \phi} = \frac{\partial X}{\partial \phi}, \frac{\partial^2 M_y}{\partial \phi^2} = \frac{\partial^2 X}{\partial \phi^2}, \frac{\partial M_y}{\partial t} = \frac{\partial X}{\partial t} + Y'(t),$$
$$\frac{\partial^2 M_y}{\partial t^2} = \frac{\partial^2 X}{\partial t^2} + Y''(t)$$

Hence equation (2.187) becomes:

$$v^2\left(\frac{\partial^2 X}{\partial r^2} + \frac{1}{r}\frac{\partial X}{\partial r} + \frac{1}{r^2}\frac{\partial^2 X}{\partial \phi^2}\right) = \frac{-\partial^2 X}{\partial t^2} + Y''(t) \qquad (2.188)$$

If we simplify the problem by assuming:

$$Y''(t) = 0,$$

Then equation (2.188) becomes:

$$v^2\left(\frac{\partial^2 X}{\partial r^2} + \frac{1}{r}\frac{\partial X}{\partial r} + \frac{1}{r^2}\frac{\partial^2 X}{\partial \phi^2}\right) = \frac{-\partial^2 X}{\partial t^2} \qquad (2.189)$$

$$\frac{\partial^2 X}{\partial r^2} + \frac{1}{r}\frac{\partial X}{\partial r} + \frac{1}{r^2}\frac{\partial^2 X}{\partial \phi^2} = -\frac{1}{v}\frac{\partial^2 X}{\partial t^2} \qquad (2.190)$$

The solution to this equation is:

$$X(r,\phi,t) = \{C_1 e^{\alpha v t} + C_2 e^{-\alpha v t}\}\{A_1 J_\beta(\alpha r) + B_1 Y_\beta(\alpha r)\}\{P\cos\beta\phi + Q\sin\beta\phi\}$$

Therefore, the general solution in this case is:

$$M_y(rC\phi, t) = \{C_1 e^{\alpha vt} + C_2 e^{-\alpha vt}\}$$
$$\{A_1 J_\beta(\alpha r) + B_1 Y_\beta(\alpha r)\}\{P\cos\beta\phi + Q\sin\beta\phi\} + \gamma(t) \qquad (2.191)$$

Case (iv)

If M_y is radially symmetric, it does not depend on ϕ and if $F_0\gamma B_1(t) = 0$, we have:

$$v^2\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r}\right) = -\frac{\partial^2 M_y}{\partial t^2}$$
$$\frac{\partial M_y}{\partial r} + \frac{1}{r}\frac{\partial M_y}{\partial r} = -\frac{1}{V^2}\frac{\partial^2 M_y}{\partial t^2} \qquad (2.192)$$

By the method of separation by variables, we have:

$$M_y = R(r)\,T(t)$$

$$\frac{\partial^2 M_y}{\partial r^2} = T\frac{\partial^2 R}{\partial r^2}, \quad \frac{\partial M_y}{\partial r} = T\frac{\partial R}{\partial r}, \quad \frac{\partial^2 M_y}{\partial t^2} = R\frac{\partial^2 T}{\partial t^2}$$

Equation (2.192) becomes:

$$T\frac{\partial^2 R}{\partial r^2} + \frac{T}{r}\frac{\partial R}{\partial r} = -\frac{R}{v^2}\frac{\partial^2 T}{\partial t^2}$$

Multiplying all through by $\frac{1}{RT}$,

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} = -\frac{1}{v^2}\cdot\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \qquad (2.193)$$

It follows that:

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} = -\frac{1}{v^2}\cdot\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\lambda^2$$

Where we have:

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} = -\lambda^2$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{R}}{\partial \mathrm{r}} + \lambda^2 R = 0 \tag{2.194}$$

and

$$-\frac{1}{\mathrm{v}^2}\cdot\frac{1}{\mathrm{T}}\frac{\partial^2 T}{\partial t^2} = -\lambda^2$$

$$\frac{\partial^2 \mathrm{T}}{\partial \mathrm{t}^2} = \lambda^2 \mathrm{v}^2 \mathrm{T} \tag{2.195}$$

The solution of equation (2.194) is given as:

$$\mathrm{R(r)} = P_2 J_0(\lambda r) + Q_2 y_0(\lambda r)$$

To solve equation (2.195), we write:

$$\mathrm{T(t)} = De^{nt}$$

Then equation (2.195) becomes:

$$n^2 = \lambda^2 \mathrm{v}^2$$

$$n = \pm \lambda v$$

Hence,

$$\mathrm{T(t)} = D_1 e^{\lambda vt} + D_2 e^{-\lambda vt}$$

Therefore,

$$M_y(r,t) = \{ D_1 e^{\lambda vt} + D_2 e^{-\lambda vt} \}\{ P_2 J_0(\lambda r) + Q_2 y_0(\lambda r)\} \tag{2.196}$$

We can then apply the boundary conditions as appropriate.

2.17 MRI Bessel Equation

We study the flow properties of the modified time independent Bloch NMR flow equations which describes the dynamics of the hydrogen atom under the influence of rf magnetic field as follows:

$$v^2 \frac{d^2 M_y}{dx^2} + T_0 v \frac{dM_y}{dx} + S(x) M_y = \frac{M_0 \gamma B_1(x)}{T_1} \tag{2.197}$$

where

$$\mathrm{S(x)} = \gamma^2 \mathrm{B}_1^2(\mathrm{x}) + \mathrm{T_g}, \quad \mathrm{T_g} = \frac{1}{\mathrm{T_1 T_2}},$$

$$\mathbf{T_0} = \frac{\mathbf{1}}{\mathbf{T_1}} + \frac{\mathbf{1}}{\mathbf{T_2}}$$

However, if the fluid velocity is dependent on x as follows:

$$v = \frac{x}{\delta} \tag{2.198}$$

where δ is the time required for the spins to cover the distance x. Since v is no longer constant, we may write:

$$v^2 \frac{d^2 M_y}{dx^2} + \left(T_0 + \frac{dv}{dx} \right) v \frac{dM_y}{dx} + S(x) M_y = \frac{M_0 \gamma B_1(x)}{T_1} \tag{2.199}$$

$$\mathrm{v}^2 \frac{\mathrm{d}^2 \mathrm{M_y}}{\mathrm{dx}^2} + \left(\mathrm{T_0} + \frac{1}{\delta} \right) \mathrm{v} \frac{\mathrm{dM_y}}{\mathrm{dx}} + \mathrm{S(x) M_y} = \frac{\mathrm{M_0} \gamma \mathrm{B_1(x)}}{\mathrm{T_1}} \tag{2.200}$$

If we design the radiofrequency field such that:

$$\gamma \mathrm{B_1(x)} = \mathrm{i} \alpha \mathrm{x}^{\mathrm{r}}, \quad \alpha = \frac{\gamma \mathrm{G} \tau}{\delta}, \quad \mathrm{r} = 1 \tag{2.201}$$

where G is the gradient magnetic field, γ is the gyromagnetic ratio and τ is the length of time for which the gradient pulse is applied and the time δ is defined as:

$$\delta = \frac{T_1 T_2}{T_1 + T_2} \tag{2.202}$$

If the NMR signal is sampled when the applied radiofrequency energy successfully displaces most of the spin unto the transverse plane ($M_0 \approx 0$), equation (2.197) then becomes:

$$v^2 \frac{d^2 M_y}{dx^2} + 2T_0 v \frac{dM_y}{dx} + \left(-\left(\frac{\gamma G \tau}{\delta} \right)^2 x^2 + \frac{1}{T_1 T_2} \right) M_y = 0 \tag{2.203}$$

$$x^2 \frac{d^2 M_y}{dx^2} + 2x \left(\frac{T_1 + T_2}{T_1 T_2} \right) \left(\frac{T_1 T_2}{T_1 + T_2} \right) \frac{dM_y}{dx} + \left(-(\gamma G \tau)^2 x^2 + \frac{1}{T_1 T_2} \left(\frac{T_1 T_2}{T_1 + T_2} \right)^2 \right) M_y = 0 \tag{2.204}$$

If we also set,

$$k = \gamma G \tau$$

and

$$\beta^2 = \frac{T_1 T_2}{(T_1 + T_2)^2} = \frac{T_g}{(T_0)^2} \tag{2.205}$$

we may therefore write:

$$x^2 \frac{d^2 M_y}{dx^2} + 2x \frac{dM_y}{dx} - \left(k^2 x^2 - \beta^2 \right) M_y = 0 \tag{2.206}$$

Equation (2.206) is an equation transformable to Bessel function [60]. Since we require that the transverse magnetization be finite as x tends to infinity, the solution is given as:

$$M_y(x) = C_1 \sqrt{x} J_n(kx) \tag{2.207}$$

where:

$$n=\frac{\sqrt{1-\beta^2}}{2} \tag{2.208}$$

Phase of the Spin

In equation (2.205), if we set:

$$\alpha = kx \tag{2.209}$$

$$\frac{dM_y}{dx}=\frac{dM_y}{d\alpha}\frac{d\alpha}{dx}=k\frac{dM_y}{d\alpha}$$

$$\frac{d^2M_y}{dx^2}=\frac{d^2M_y}{d\alpha^2}\left(\frac{d\alpha}{dx}\right)^2+\frac{dM_y}{d\alpha}\frac{d^2\alpha}{dx^2}=k^2\frac{d^2M_y}{d\alpha^2}$$

Equation (2.206) becomes:

$$k^2x^2\frac{d^2M_y}{d\alpha^2}+2kx\frac{dM_y}{d\alpha}+\left(k^2x^2+\beta^2\right)M_y=0$$

$$\alpha^2\frac{d^2M_y}{d\alpha^2}+2\alpha\frac{dM_y}{d\alpha}+\left(\alpha^2+\beta^2\right)M_y=0 \tag{2.210}$$

We shall make another transformation as follows:

$$M_y=\frac{U}{\alpha} \tag{2.211}$$

$$\begin{aligned}\frac{dM_y}{d\alpha}&=\frac{1}{\alpha}\frac{dU}{d\alpha}-\frac{U}{\alpha^2}\\ \frac{d^2M_y}{d\alpha^2}&=\frac{1}{\alpha}\frac{d^2U}{d\alpha^2}-\frac{2}{\alpha^2}\frac{dU}{d\alpha}+\frac{2U}{\alpha^3}\end{aligned} \tag{2.212}$$

Hence, equation (2.211) becomes:

$$\alpha^2\left(\frac{1}{\alpha}\frac{d^2U}{d\alpha^2}-\frac{2}{\alpha^2}\frac{dU}{d\alpha}+\frac{2U}{\alpha^3}\right)+2\alpha\left(\frac{1}{\alpha}\frac{dU}{d\alpha}-\frac{U}{\alpha^2}\right)+\left(\alpha^2+\beta^2\right)\frac{U}{\alpha}=0$$

$$\left(\alpha\frac{d^2U}{d\alpha^2}-2\frac{dU}{d\alpha}+\frac{2U}{\alpha}\right)+\left(2\frac{dU}{d\alpha}-\frac{2U}{\alpha}\right)+\left(\alpha^2+\beta^2\right)\frac{U}{\alpha}=0$$

$$\alpha\frac{d^2U}{d\alpha^2}+\left(\alpha^2+\beta^2\right)\frac{U}{\alpha}=0 \tag{2.213}$$

$$\frac{d^2U}{d\alpha^2}+\left(1+\frac{\beta^2}{\alpha^2}\right)U=0 \tag{2.214}$$

Based on the Short Gradient Pulse (SGP) approximation [66], the parameter α represents the phase of the spin such that the effect of a gradient pulse of duration τ on a spin at position x is given by, neglecting the effect of the static field,

$$\alpha(x)=\gamma G\tau x=\phi(x) \tag{2.215}$$

Hence, if we consider the phase change of a spin which was at position χ_0 during the first gradient pulse and at position χ_1 during the second, then the change in phase in moving from χ_0 to χ_1 is given by

$$\Delta\alpha(x_1-x_0)=\gamma G\tau(x_1-x_0) \tag{2.216}$$

Therefore, we see that:

$$\frac{\beta^2}{\alpha^2}=\frac{T_g}{(\phi(x)T_0)^2} \tag{2.217}$$

2.18 Equation of Motion for Pulsed NMR

In a typical MRI procedure where G is the pulsed gradient applied for the length of time τ, equation (2.40) becomes:

$$x^2\frac{d^2M_y}{dx^2}+x\sigma\frac{dM_y}{dx}+\left(\left(\gamma\frac{\sigma}{T_o}G\right)^2x^2+\frac{\sigma^2}{T_1T_2}\right)M_y=\frac{M_o\gamma B_1(x)}{T_1} \tag{2.218}$$

where

$$\gamma B_1(x)=\gamma Gx$$

and

$$v=xT_o=\frac{x}{\tau}$$

For 90^o pulse M_o is minimum (say $M_o = 0$), we can write equation (2.218) as:

$$x^2\frac{d^2M_y}{dx^2}+x\sigma\frac{dM_y}{dx}+\left((\gamma G\tau)^2x^2+\frac{\tau^2}{T_1T_2}\right)M_y=0 \tag{2.219a}$$

$$x^2\frac{d^2M_y}{dx^2}+x\sigma\frac{dM_y}{dx}+\left((\theta)^2x^2+\frac{\tau^2}{T_1T_2}\right)M_y=0 \tag{2.219b}$$

where

$$\frac{1}{T_o}=\tau \tag{2.220}$$

Equation (2.219) is a general form of an equation transformable into Bessel equation of order β with parameter k. In equation (2.219), the flip angel is defined as:

$$\theta(rad)=\gamma(rad\ \ \sec^{-1}\ G^{-1})(G)\tau(\sec) \tag{2.221}$$

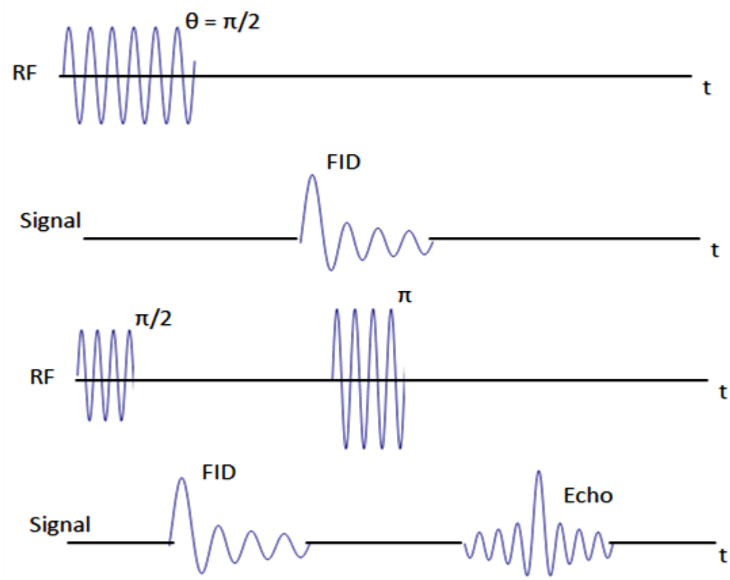


Fig. 2.1 *The NMR Pulse sequence.*

Equation (2.219) presents new ways of generating the NMR signal using the Bessel functions. As shown in figure 1, the experiment is started with a $\frac{\pi}{2}$ pulse, following which the magnetic vector M_y precesses in the plane perpendicular to the direction of static magnetic field B_o and free induction decay (FID) occurs. The maximum amplitude of the FID is measured to obtain a voltage-amplitude. After a delay which is typically of the order of 10ms, a π rf pulse is introduced. Following another interval τ, the magnetic spins recluster and a spin echo voltage signal is observed. The voltage amplitude of the spin echo is taken as proportional to $M_{y,}$ equation (2.219) is then solved. To solve equation (2.219b), the value of G must be known, as well as the gyromagnetic ratio γ of the specific nuclei under study. The voltage amplitude of the spin echo M_y is easily computed by solving the Bessel equation of order β and parameter θ as shown in equation (2.219b) where:

$$\beta = \tau\sqrt{T_g} \tag{2.222a}$$

and

$$\theta = \gamma G \tau \tag{2.222b}$$

2.19 Application to Molecular Imaging

In this section, new magnetic resonance methodology to solve the Bloch NMR flow equation based on Hermite, Bessel and simple quantum mechanical functions for detailed studies of processes taking place at the molecular level in living tissues have been developed. We show how these quantum mechanical functions may be very crucial in the assessment of Cancer cells, Multiple sclerosis (MS) and Brain edema using magnetic resonance imaging.

We apply a fundamental transformation procedure on equation (2.40) given by:

$$\mathrm{M_y(x) = \psi(x)e^{\lambda x}} \tag{2.223a}$$

provided that:

$$\lambda = -\frac{1}{2vT_0}, \tag{2.223b}$$

Equation (2.40) becomes:

$$\frac{d^2\psi(x)}{dx^2} + \frac{1}{v^2}\left(T_g - T_R - \gamma^2 G^2 x^2\right)\psi(x) = 0; \tag{2.224a}$$

where

$$T_g = \frac{1}{T_1 T_2}; \; T_R = \frac{1}{4T_0^2} \tag{2.224b}$$

and G is the strength of the gradient field. With further assumptions:

$$\gamma B_1(x) = i\gamma G x; \; \alpha = i\sqrt{\frac{\gamma G}{v}}x; \tag{2.225}$$

we obtain:

$$\frac{d^2\psi}{d\alpha^2} + \left(\frac{T_g - T_R}{\gamma G v} - \alpha^2\right)\psi = 0. \tag{2.226}$$

If we make another transformation as follows:

$$\psi(\alpha) = M_y(\alpha) e^{-\frac{\lambda}{i}\sqrt{\frac{v}{\gamma G}}\alpha} \text{ and } \frac{\lambda}{i}\sqrt{\frac{v}{\gamma G}} = \frac{\zeta\alpha}{2}, \tag{2.227}$$

we have:

$$\frac{d^2 M_y}{d\alpha^2} - 2\alpha\frac{dM_y}{d\alpha} + 2nM_y = 0 \tag{2.228}$$

and given that:

$$\left.\begin{aligned} & T_g - T_R = (2n+1)\gamma G v \\ & \zeta \approx 1 \end{aligned}\right\}, \tag{2.229}$$

the final solution becomes:

$$M_y(x) = H_n\left(i\sqrt{\frac{\gamma G}{v}}x\right) = (-1)^n \exp\left(-\frac{\gamma G}{v}x^2\right)\frac{d^n}{dx^n}\exp\left(-\frac{\gamma G}{v}x^2\right) \tag{2.230a}$$

where

$$n = \frac{T_R - T_g}{2\gamma G v} - \frac{1}{2} \tag{2.230b}$$

From equations (2.222, 2.230), the term $\frac{1}{\zeta}$ represents the phase change of the spin at the position x, provided that the relaxation times T_1 and T_2 are properly chosen to represent the gradient pulse duration in the pulsed-field gradient (PFG) NMR as shown in figures 2.2-2.3.

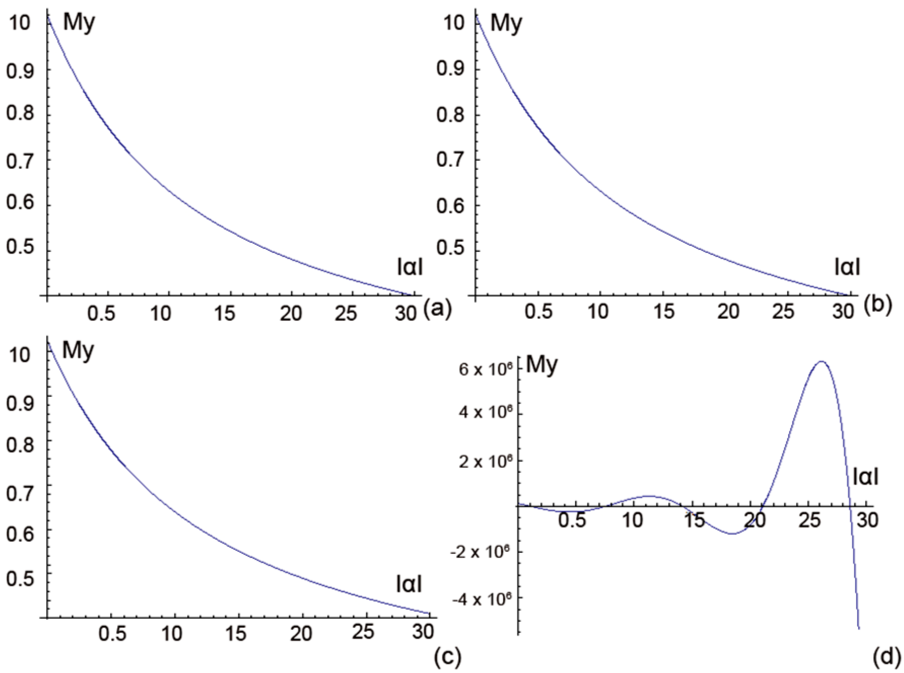


Fig. 2.2 *Plots of the transverse magnetization M_y against the absolute (positive) values of α using the relaxations time – values of kidney at 1.5T [3], G = 10mT/m, γ = 42.5781 x 10^6/T/s for (a) v = 3.0m/s (b) v = 0.3m/s (c) v = 0.003m/s (d) v = 0.000003m/s.*

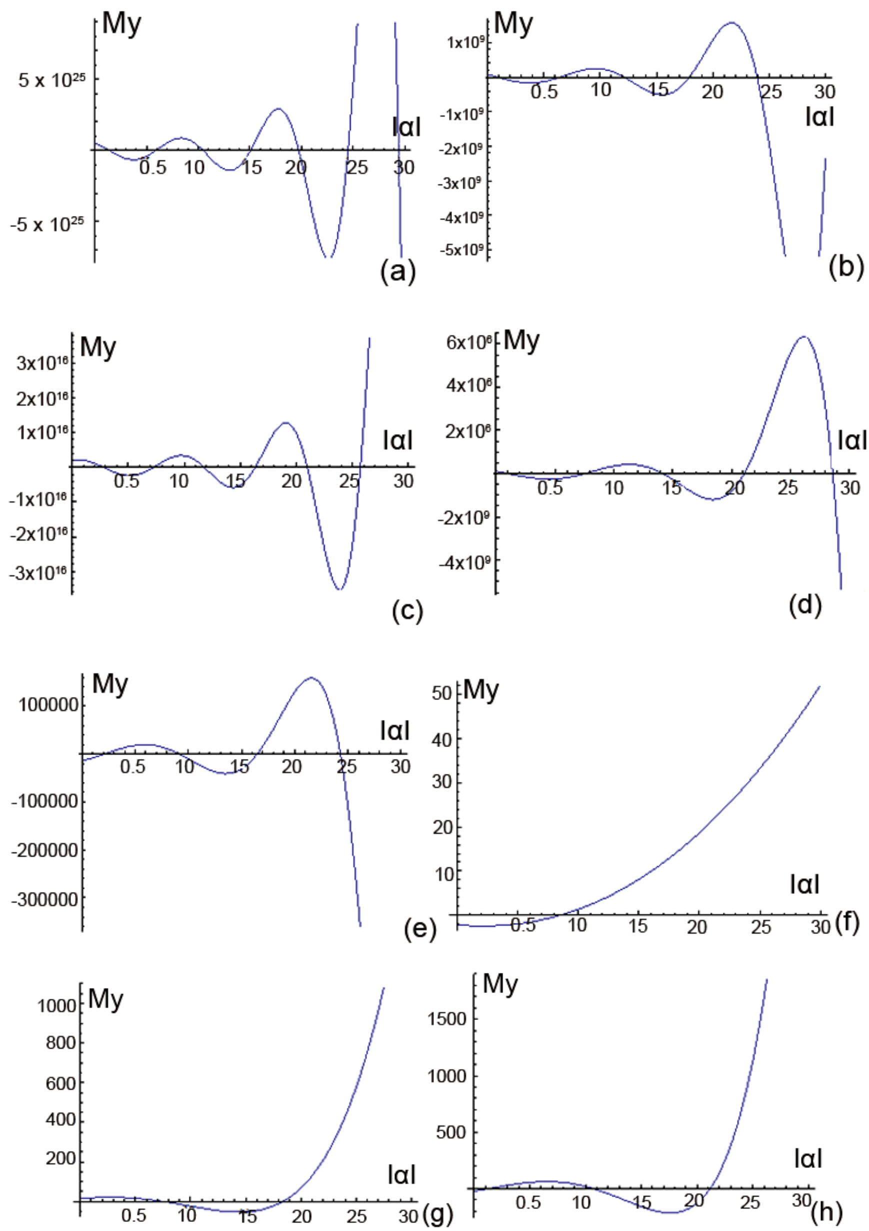


Fig. 2.3 *Plots of the transverse magnetization M_y against the absolute (positive) values of α; $G = 10mT/m$, $\gamma = 42.5781 \times 10^6/T/s$, $v = 0.000003m/s$ using the relaxations time – values, at 1.5T [3], of (a) skeletal muscle (b) heart muscle (c)liver (d) kidney (e) spleen (f) fatty tissue (g) gray brain matter (h) white brain matter.*

It is observed from figures (2.1) and (2.2) that as the fluid velocity reduces as often encountered in cellular processes, the imaging equation as given in equation (2.224) shows contrast in terms of MR signals. Figure (2.2) shows that the behaviour of the MR signals is completely different for different tissues. It is quite interesting to note that the magnitude of the signals becomes so large at the molecular level. This can make it possible to follow processes at molecular level in real time with brighter images.

2.20 The Hermite Polynomials

Based on equations (2.224-2.225), equation (2.40) becomes:

$$\frac{d^2\psi(x)}{dx^2} + \left(\alpha - \beta^2 x^2\right)\psi(x) = 0 \tag{2.231}$$

$$\alpha = \frac{T_g - T_R}{v^2} \; : \; \beta = \frac{\gamma G}{v}; \; T_g = \frac{1}{T_1 T_2} \text{ and } T_R = \frac{1}{4T_0^2}.$$

where G is the strength of the gradient field, The solutions of equation (2.40) are shown in table 1;

Table 1. *Solution of the Bloch NMR flow equation in terms of Hermite Polynomials.*

n	$\left\lvert\frac{(T_G - T_R)}{v\gamma G}\right\rvert$	M_{yn}
0	1	$\left(\frac{\gamma G}{\pi v}\right)^{\frac{1}{4}} e^{-\left(\frac{x}{2vT_o}+\frac{\gamma G x^2}{2v}\right)}$
1	3	$\frac{1}{\sqrt{2}}\left(\frac{\gamma G}{\pi v}\right)^{\frac{1}{4}}\left(2\sqrt{\frac{\gamma G}{v}}x\right)e^{-\left(\frac{x}{2vT_o}+\frac{\gamma G x^2}{2v}\right)}$
2	5	$\frac{1}{\sqrt{8}}\left(\frac{\gamma G}{\pi v}\right)^{\frac{1}{4}}\left(\frac{4\gamma G x^2}{v}-2\right)e^{-\left(\frac{x}{2vT_o}+\frac{\gamma G x^2}{2v}\right)}$
3	7	$\frac{1}{\sqrt{48}}\left(\frac{\gamma G}{\pi v}\right)^{\frac{1}{4}}\left(8\left(\frac{\gamma G}{v}\right)^{\frac{3}{2}}x^3 - 12\sqrt{\frac{\gamma G}{v}}x\right)e^{-\left(\frac{x}{2vT_o}+\frac{\gamma G x^2}{2v}\right)}$

where: $\frac{\alpha}{\beta} = (2n+1)$

$$\text{or } \left\lvert\frac{(T_G - T_R)}{(2n+1)\gamma G}\right\rvert = v \text{ and n = 0, 1, 2, 3, 4, 5,.........} \tag{2.232}$$

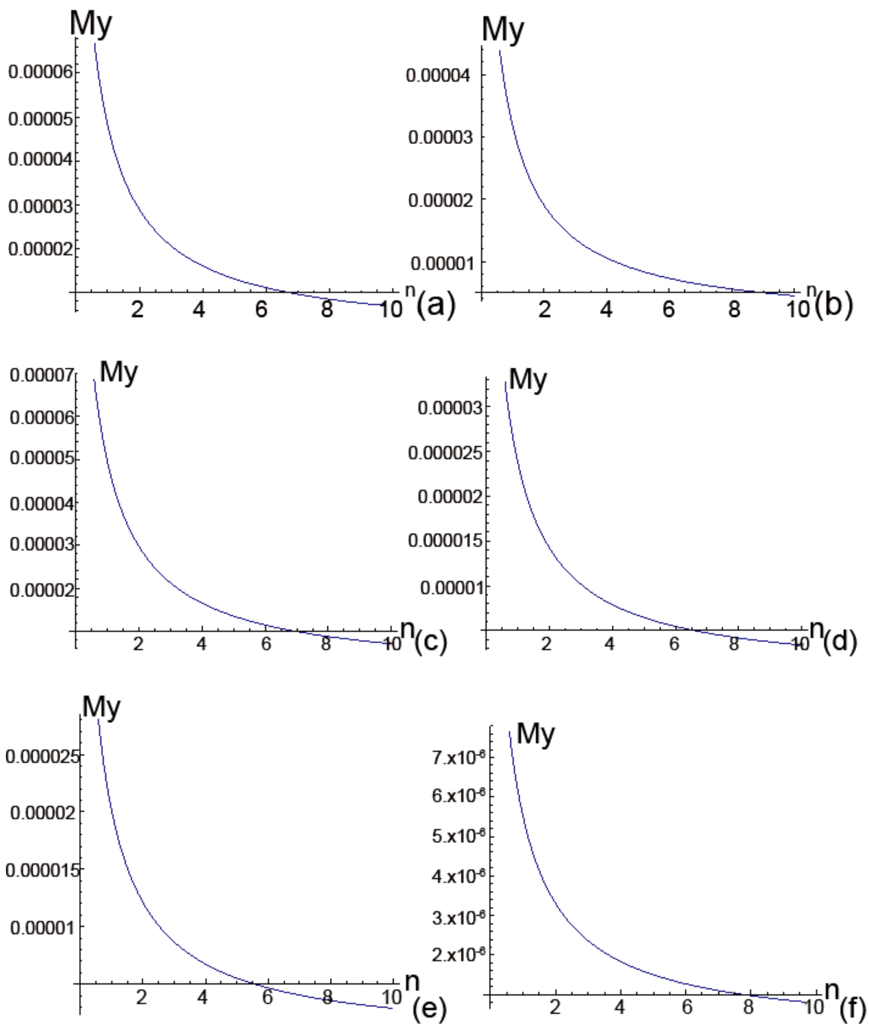


Fig. 2.4 *Plots of fluid velocity against parameter n when $G = 10mTm^{-1}$, $\gamma = 42.5781 \times 10^6 T^{-1}s^{-1}$ using the relaxations time – values, at 1.5T [3], of (a) skeletal muscle (b) heart muscle (c)liver (d) kidney (e) spleen (f) fatty tissue (g) gray brain matter (h) white brain matter.*

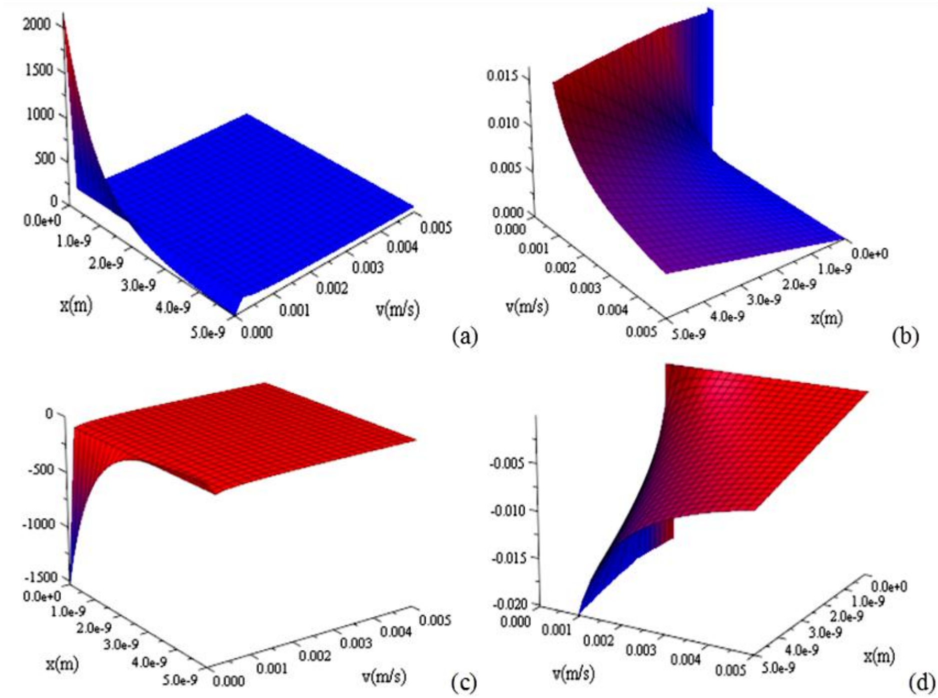


Fig. 2.5 *Plots of transverse magnetization against v and x; G = 10mTm^{-1}, γ = 42.5781 x 10^{6}T^{-1}s^{-1} using the relaxations time – values of kidney at 1.5T [2] for (a) n = 0 (b) n = 1(c) n = 2 (d) n = 3.*

Based on equation (2.232) and figure (2.3), the highest velocity is recorded when n = 0 and the velocity decreases when n increases. For each value of n, the NMR signal at the molecular level is obtained as Based on equation (2.232) and figure (2.4), the highest velocity is recorded when n = 0 and the velocity decreases when n increases. For each value of n, the NMR signal at the molecular level is obtained as shown in table 1 and equation (2.218). If Ax^2 is defined as the cross sectional area of blood vessels, the method can be very useful in estimating blood flow in capillaries or veins at the molecular level, especially in the assessment of angiogenesis and cancer proliferation. It is worthy of note that as the values of n increases (moderate decrease in the fluid velocity), the signal behaves like that of electron spin resonance (Figs 2.5c and

2.5d). This may be very important in the imaging of the complex molecular changes often observed in cancer cells.

2.21 Application to Multiple Sclerosis

Multiple sclerosis (MS) is an inflammatory disease in which the fatty myelin sheaths around the axons of the brain and spinal cord are damaged, leading to demyelination and scarring as well as a broad spectrum of signs and symptoms. MS affects the ability of nerve cells in the brain and spinal cord to communicate with each other effectively. Nerve cells communicate by sending electrical signals called action potentials down long fibers called axons, which are contained within an insulating substance called myelin. In MS, the body's own immune system attacks and damages the myelin. When myelin is lost, the axons can no longer effectively conduct signals [67]. Although much is known about the mechanisms involved in the disease process, the cause remains unknown.

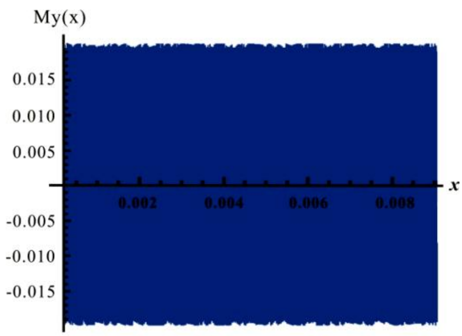
MRI has been considered to be the most informative noninvasive method to diagnose and monitor disease progression in patients with multiple sclerosis (MS) [68]. However, conventional T_2-weighted MR images do not sufficiently correlate with histo-pathological substrates and clinical disability [68]. Conventional T_2-weighted images are unable to distinguish underlying histo-pathological substrates, such as inflammation, edema, demyelination, gliosis, and axonal loss, because all of these lesions have identical high signal on T_2-weighted images. We study equation (2.206) when the transverse magnetization is finite as x tends to infinity. The solution is given as:

$$M_y(x) = C_1\sqrt{x}J_n(kx) \tag{2.233a}$$

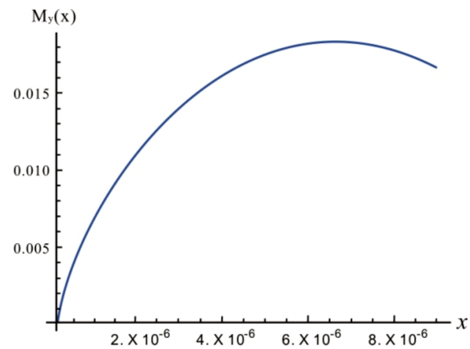
where:

$$n = \frac{\sqrt{1-\beta^2}}{2} \tag{2.233b}$$

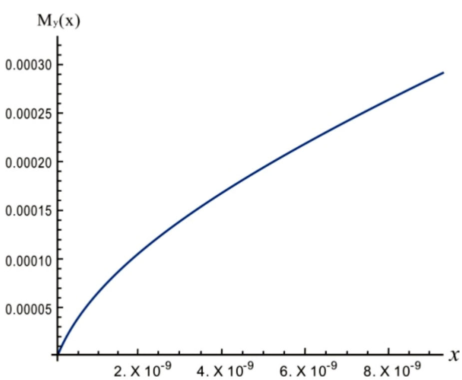
Since the abnormalities observed in MS mostly involve the white matter ($T_1 = 0.78s$, $T_2 = 0.09s$), gray matter ($T_1 = 0.92s$, $T_2 = 0.10s$) and CSF ($T_1 = 4.50s$, $T_2 = 2.30s$), we shall make use of their relaxation properties at 1.5T to compare their transverse magnetization for different ranges of x as shown in figure 2.5.



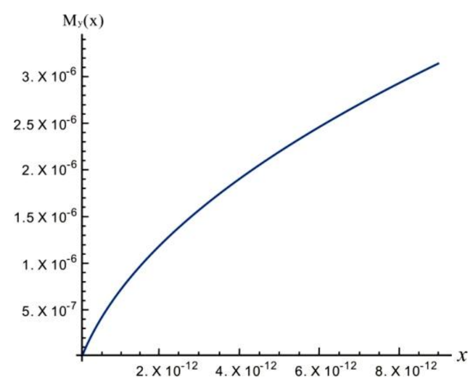
(a1)



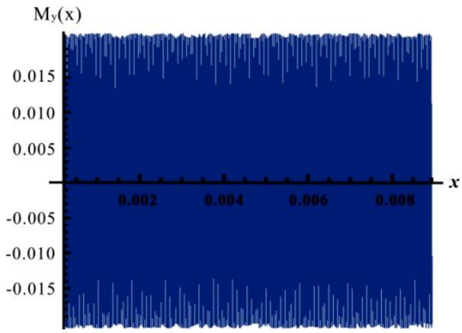
(a2)



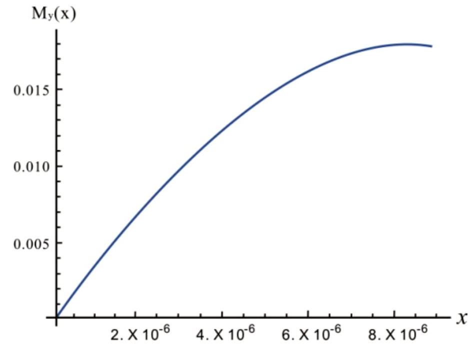
(a3)



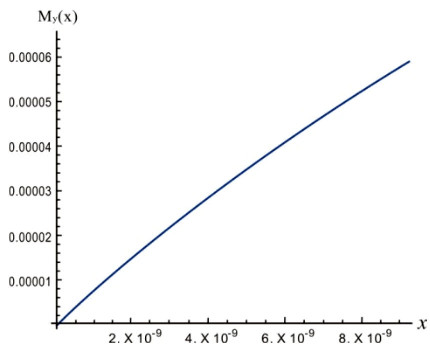
(a4)



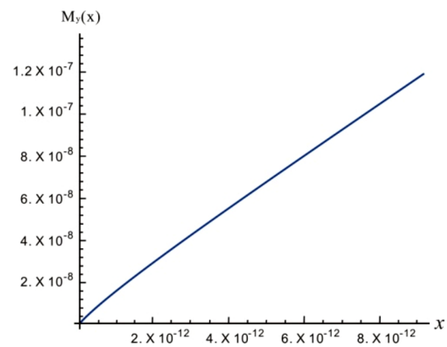
(b1)



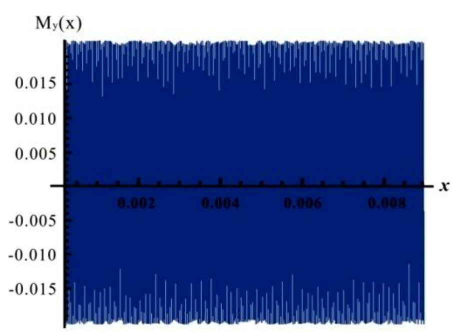
(b2)



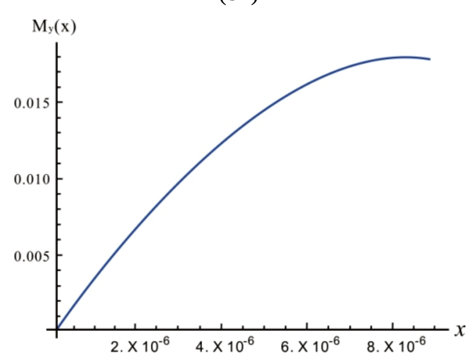
(b3)



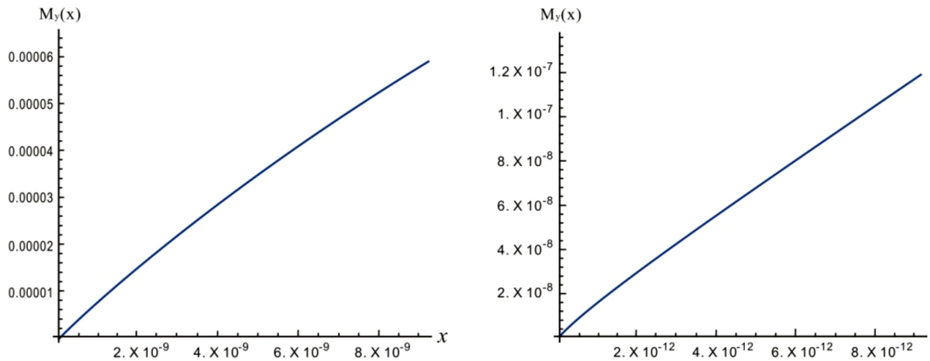
(b4)



(c1)



(c2)



(c3) (c4)

Fig. 2.6 *Plots of transverse magnetization against x for $C_1 = 10$, $G = 0.033T/m$ and $\tau = 0.02s$; using the relaxation times of (ai) CSF in milli range (a2) CSF in micro range (a3) CSF in nano range (a4) CSF in pico range (bi) Gray matter in milli range (b2) Gray matter in micro range (b3) Gray matter in nano range (b4) Gray matter in pico range (ci) White matter in milli range (c2) White matter in micro range (c3) White matter in nano range (c4) Whiet matter in pico range.*

The results obtained in Fig. 2.6 confirm what has been observed in T_2 – weighted MR experiments. Changes in relaxation times that are direct indication of histo-pathological substrates do not contribute significantly to the magnitude of the MR signal. That is, dynamics of these important substrates cannot easily be seen on MR scans. However, from figure 2.6, we have observed that the transverse magnetization is actually responding very slowly to small changes in T_0. The magnitude of M_y is changing very slowly from one tissue to another. Secondly, we see that whenever x is in the microscopic range, the behavior of M_y changes uniquely and since most tissue processes are found in this geometrical range, inflammation, edema formation, demyelination, gliosis and axonal may easily be imaged. However, to realize this, the MR algorithm must be designed such that C_1 takes on very high values in order to improve signal magnitudes. Finally, we suggest that higher static field strength may be required for good MR assessment of multiple sclerosis, although the influence of such high fields and the associated gradient fields on normal tissue functions must first be taken into consideration.

2.22 Bloch - Torrey Equation for NMR Studies of Molecular Diffusion

Since the diffusion coefficient varies very slowly with the radial distance r, it is interesting to note that $B_1(x,t)$ in equation (2.33) can be defined appropriately based on the problems to be solved. For example if we define $B_1(x,t)$ as:

$$\gamma B_1(r,t) = f(r) M_y(r,t) \tag{2.234}$$

Parameter f(r) in equations (2.234) can be appropriately defined to solve specific biological and medical problems. Generally, equation (2.33) becomes:

$$\frac{\partial M_y}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial M_y}{\partial r}\right) + F_0 f(r) M_y \tag{2.235}$$

The Bloch –Torey equation for the magnetization density $M_y(r,t)$ arising from spins diffusing with diffusion coefficient D and an arbitrary time-dependent linear gradient field is obtained from equation (2.33) if we define $B_1(x,t)$ as:

$$\gamma B_1(r,t) = -igf(t) M_y(r,t) \tag{2.236}$$

where g denotes the product of F_o and the gradient strength, the gradient field has the temporal shape function f(t) in the direction x, where r is is the position vector of the spin. For one dimensional motion, equations (2.33) becomes:

$$\frac{\partial M_y}{\partial t} = D\frac{\partial^2 M_y}{\partial x^2} - igf(t) M_y(x,t) \tag{2.237}$$

Equation (2.237) is the Bloch-Torrey equation which has been solved for the NMR study of molecular diffusion [69, 70].

2.23 Adiabatic Model of Bloch NMR Flow Equation

In this section, we consider equation (2.25) under adiabatic condition when the rf B_1 field is a constant. The adiabatic condition is defined as:

$$\gamma^2 B_1^2(x) >> \left(\frac{1}{T_1 T_2}\right) \tag{2.238}$$

Based on equation (2.238), equation (2.25) can be written as:

$$v^2 T_1^2 \frac{d^2 M_y}{dx^2} + v T_1^2 \left(\frac{1}{T_1} + \frac{1}{T_2}\right) \frac{dM_y}{dx} + \varsigma^2 M_y = \varsigma M_o \tag{2.239}$$

where

$$vT_1 = \sqrt{1 - x^2} \quad 0 < x < 1 \ \text{ for real value of } v \tag{2.240}$$

$$v = \frac{-4xT_1 T_2}{T_1^2 (T_1 + T_2)} \ \text{when n odd n} = 1, 3, 5, 7, 9, \ldots \ldots \tag{2.241}$$

$$\varsigma = T_1 \gamma B_1 \tag{2.242}$$

For human blood flow of T_1 = 1.0s, the parameter ς is a real constant which completely defines constant values of rf B_1 field for the NMR system [71, 72].

$$\varsigma = \gamma B_1 \tag{2.243}$$

The application of equation (2.239-2.242) has been demonstrated [71].

2.24 Application of Time Dependent Bloch NMR Equation and Pennes Bioheat Equation to Theranostics

Theranostics is the combination of therapeutics and diagnostics. It has been regarded as a key part of personalized medicine and requires considerable advances in predictive medicine; novel theranostic agents are developed and carefully designed for in vivo quantitative assessment of the amount of drug reaching a pathological region and the visualization of molecular changes due to the therapeutic effects of the delivered drug. This study intends to mathematically model a closely knitted theranostic method in which a specially selected radiofrequency field is used to heat up a tissue and at the same time cause the spins of the tissue to emit MR signals.

The key to this application is the specific absorption rate (SAR) which drives both rf power heating within a tissue and is related directly to the B_1 field which is needed to cause spin resonance. If we consider bioheat flow in one direction [73, 42] with uniform tissue properties, we have:

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + SAR - w_b c_b (T - T_b) \tag{2.244}$$

where ρ is tissue density, *c* is the specific heat of tissue, *T* is the tissue temperature, *t* is the time, w_b is the blood perfusion rate, c_b is the specific heat of blood, T_b is the supplying arterial blood temperature, *k* is the thermal conductivity of tissue, and *x* is the distance from the skin surface. SAR is the applied rf power per unit volume. If the tissue temperature changes very slowly with x, we have:

$$\rho c \frac{\partial T}{\partial t} = SAR - w_b c_b (T - T_b) \tag{2.245}$$

The solution to equation (245) is given as follows:

$$T(t) = T_b + \frac{SAR}{w_b c_b} + A \exp\left(-\frac{w_b c_b}{\rho c} t \right) \tag{2.246}$$

If the tissue temperature before the application of the rf field does not defer significantly from the arterial temperature, the initial the condition for this problem is given as:

$$T(t=0)=T_b \tag{2.247}$$

We finally have:

$$T(t)=T_b+\frac{SAR}{w_b c_b}\left\{1-\exp\left(-\frac{w_b c_b}{\rho c}t\right)\right\} \tag{2.248}$$

SAR and Oscillating Magnetic Field: The rf power for the voxel volume V_{vox} is P_{rf} = *(SAR)* V_{vox} . The energy of the oscillating radio wave is given as $E_{rf} = \hbar\gamma B_1$, whose rate of change is:

$$P_{rf}=\frac{dE_{rf}}{dt} \text{ and } E_{rf}=\int_{t_0}^{t}(SAR)V_{vox}dt \tag{2.249}$$

$$\gamma B_1(t)=\frac{V_{vox}}{\hbar}(SAR)(t-t_0) \tag{2.250}$$

SAR and Time dependent NMR Equation: We can relate time dependent MRI signal to SAR using the time independent NMR equation [42] given by equation (2.251) and (2.252):

$$\frac{d^2M_y}{dt^2}+T_o\frac{dM_y}{dt}+\left(T_g+\gamma^2B_1{}^2(t)\right)M_y=\frac{M_o}{T_1}\gamma B_1(t) \tag{2.251}$$

$$\text{where } T_g=\frac{1}{T_1T_2} \text{ and } T_0=\frac{1}{T_1}+\frac{1}{T_2} \tag{2.252}$$

If we sample the signal when the transverse component of the magnetization has the largest amplitude, we write $M_0 \approx 0$. Provided that the condition $T_g \ll \gamma^2 B_1^2(t)$ holds, equation (2.251) becomes [42]:

$$\frac{d^2M_y}{dt^2}+T_o\frac{dM_y}{dt}+\gamma^2B_1{}^2(t)M_y=0 \tag{2.257}$$

From equations (2.251) and (2.252), we obtain:

$$\frac{d^2M_y}{dt^2}+T_o\frac{dM_y}{dt}+\left(\frac{V_{vox}}{\hbar}\right)^2(SAR)^2(t-t_0)^2M_y=0 \tag{2.258}$$

If the rf B_1 field is applied at time $t_0 = 0$, we have:

$$M_y(t)=(\beta)^n(t)^n\left[C_1J_n\left(\frac{V_{vox}}{2\hbar}t^2\right)+C_2Y_n\left(\frac{V_{vox}}{2\hbar}t^2\right)\right];\ n=\frac{1-T_0t}{2} \tag{2.259}$$

This solution is valid for $T_0(t-t_0)\le 1$. It is always required that the transverse magnetization be finite as time tends to infinity, hence, we write:

$$M_y(t)=C_1(\beta)^{\frac{1-T_0t}{2}}(t)^{\frac{1-T_0t}{2}}J_{\frac{1-T_0t}{2}}\left(\frac{V_{vox}}{2\hbar}t^2\right) \tag{2.260}$$

The results obtained in equations (2.248, 2.260) have been simulated with relaxation parameters of human liver at 1.5T [74] and the corresponding thermal properties [72, 73]: T_1 = 0.610s, T_2 = 0.057s, w_b = 2.86kg/m^{3}s, c_b = 3960J/kg.K, ρ = 1060kg/m^3, c = 3600J/kg.K. Plots (a) and (b) (SAR = 4W/m^3) give the distribution of the tissue temperature and transverse magnetization on a log scale while plot (c) (SAR = 40000W/m^3) gives the density plot of the transverse magnetization as a function of time and tissue temperature.

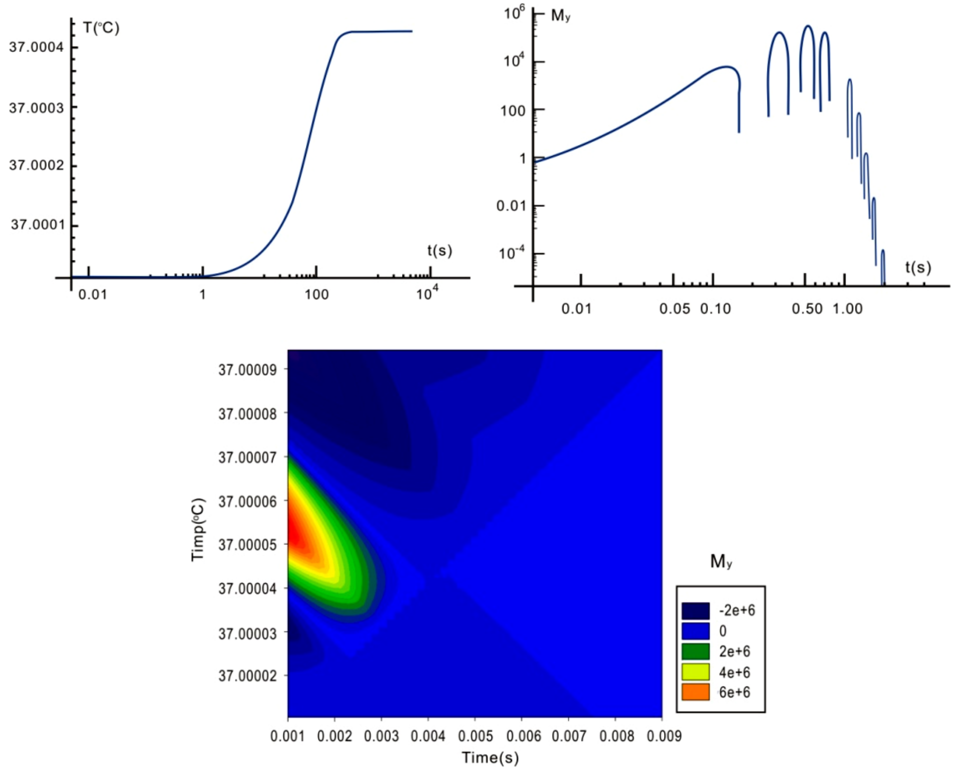


Fig. 2.7 *show (a) Temperature profile (b) Transverse magnetization profile (c) density image. This shows that provided the conditions which led to equation (2.248) are met, real time theranostic imaging can easily be done with equations (2.248, 2.260).*

The temperature distribution and the rf power needed to generate rf $B_1(t)$ field within the medically acceptable SAR limit during MRI scanning procedure have been investigated by solving the Pennes Bioheat equation in terms of MRI parameters. The relationship between temperature, SAR and rf $B_1(t)$ at any given time (under the assumed conditions) is clearly shown in equations (2.248, 2.260) and Fig.2.7. a, b, c. With this, accurate estimate of the amount of rf field needed for a particular power deposition for safe imaging of different tissues can be done. What is most interesting about the results in this study is that time, SAR and voxel volume can easily be used to manipulate the range of temperature needed for therapy without unnecessarily increasing the tissue temperature. These changes directly influence the NMR signal as clearly illustrated in Fig, 2.7c and shows that we can do tissue imaging and temperature mapping at the same time.

2.25 Summary

The major contributions of this book can be seen at a glance by the development of the following differential equations derived from the Bloch NMR flow equations. These differential equations can be referred to as the Awojoyogbe-Bloch NMR flow equations.

$$\begin{aligned} &v^2\frac{\partial M_y}{\partial x^2}+2v\frac{\partial^2 M_y}{\partial x\partial t}+v\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial x} \\ &+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial t}+\frac{\partial^2 M_y}{\partial t^2}+\left(\frac{1}{T_1T_2}+\gamma^2 B_1{}^2(x,t)\right)M_y \\ &=\frac{\gamma B_1(x,t)M_o}{T_1} \end{aligned} \tag{S1}$$

$$\begin{aligned} &\frac{d^2M_y}{dx^2}+\frac{1}{v}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{dM_y}{dx} \\ &+\frac{1}{v^2}\left(\gamma^2B_1^2(x)+\frac{1}{T_1T_2}\right)M_y \\ &=\frac{M_o\gamma B_1(x)}{v^2T_1} \end{aligned} \tag{S2}$$

$$\frac{d^2\psi}{dx^2}+\frac{1}{v^2}\left(\gamma^2B_1^2+\frac{1}{T_1T_2}\right)\psi=0 \tag{S3}$$

$$\frac{d^2M_y}{dt^2}+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{dM_y}{dt}+\left(\gamma^2B_1^2(t)+\frac{1}{T_1T^2}\right)M_y=\frac{M_0}{T_1}\gamma B_1(t) \tag{S4}$$

$$v^2\frac{\partial^2 M_y}{\partial x^2}+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial t}=0 \tag{S5}$$

$$D\frac{\partial^2 M_y}{\partial x^2}+\frac{\partial M_y}{\partial t}=0$$

where

$$D=\frac{v^2}{T_o} \tag{S6}$$

$$v^2\frac{\partial^2 M_y}{\partial x^2}+2v\frac{\partial^2 M_y}{\partial x\partial t}+\frac{\partial^2 M_y}{\partial t^2}=0 \tag{S7}$$

$$\frac{d^2 M_y}{dt^2}+T_o\frac{dM_y}{dt}+\left(T_g+\gamma^2 B_1^{\,2}(t)\right)M_y=\frac{M_o}{T_1}\gamma B_1(t) \tag{S8}$$

$$i\hbar\frac{\partial M_y}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2 M_y}{\partial x^2}+\frac{i\hbar F_o}{T_o}\gamma B_1(x,t) \tag{S9}$$

$$\left(1-\varsigma^2\right)\frac{d^2 M_y}{d\varsigma^2}-2\varsigma\frac{dM_y}{d\varsigma}+n(n+1)M_y=0 \tag{S10}$$

$$\frac{d^2 M_y}{d\alpha^2}-2\alpha\frac{dM_y}{d\alpha}+2nM_y=0 \tag{S11}$$

$$\frac{d^2\psi(x)}{dx^2}+\left(\alpha-\beta^2x^2\right)\psi(x)=0 \tag{S12}$$

$$x^2\frac{d^2M_y}{dx^2}+x\sigma\frac{dM_y}{dx}+\left(\left(\gamma\frac{\sigma}{T_o}G\right)^2x^2+\frac{1}{T_1T_2}\left(\frac{\sigma}{T_o}\right)^2\right)M_y=0 \tag{S13}$$

$$x^2\frac{d^2M_y}{dx^2}+x\sigma\frac{dM_y}{dx}+\left((\gamma G\tau)^2x^2+\frac{\tau^2}{T_1T_2}\right)M_y=0 \tag{S14}$$

$$x^2\frac{d^2M_y}{dx^2}+x\sigma\frac{dM_y}{dx}+\left((\gamma GnTR)^2x^2+\frac{(nTR)^2}{T_1T_2}\right)M_y=0 \tag{S15}$$

$$x^2 \frac{d^2 M_y}{dx^2} + x\sigma \frac{dM_y}{dx} + \left((\theta)^2 x^2 + \frac{\tau^2}{T_1 T_2} \right) M_y = 0 \tag{S16}$$

$$v^2 T_1^2 \frac{d^2 M_y}{dx^2} + v T_1^2 \left(\frac{1}{T_1} + \frac{1}{T_2} \right) \frac{dM_y}{dx} + \varsigma^2 M_y = \varsigma M_o \tag{S17}$$

$$v^2 \frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_y}{\partial t^2} = F_o \gamma B_1 (x,t) \tag{S18}$$

$$\frac{\partial M_y}{\partial t} = D \frac{\partial^2 M_y}{\partial x^2} - igf(t) M_y (x,t) \tag{S19}$$

The ideal approach to exhaust most of the quantitative and qualitative information for studying biological systems at the macroscopic and microscopic levels by magnetic resonance imaging technique with particular reference to the theory, dynamics and applications of MRI would be to find generalized (time dependent and time independent) analytical solutions to these equations. The advantages of such solutions are related to the fact that the magnetizations and signals obtainable from them are synthesis of many parameters that are of clinical importance for most magnetic resonance imaging analyses.

Solutions to these equations will result to new developments in MRI physics. Quantitative and computational analyses, mathematical modeling and analytical solutions of these equations can lead to breath taking innovations and novel applications of MRI for improved health care. High quality and novel contributions related to biological, biomedical, clinical, geophysical and any other exciting applications are welcomed in the next volume of this book. All proposals can be addressed to the editor at abamidele@futminna.edu.ng or awojoyogbe@yahoo.com.

2.26 Conclusion

In this chapter, we have modeled the Bloch NMR flow equations into Bessel equation, Diffusion equation, Wave equation, Schrödinger's equation, Legendre's equation, Euler's equation and Boubaker polynomials. While the detailed analytical solutions of the time dependent NMR flow equation and the NMR wave equation are presented, solutions to other several equations that may be derived from equation (2.18) are available in standard textbooks on physics, mathematics and engineering mathematics. With appropriate initial and boundary conditions, solutions to these equations can be applied to solve most problems that may enhance the theory, dynamics and applications of MRI. This may open a large window of opportunities for researchers in all research fields to contribute to this high intellectually adventurous field thereby improving the image quality with better treatment of deceases at the most affordable cost. It is hoped that due to the ability of magnetic resonance imaging to probe right to the fundamental level, scientists may be able to image human cellular functions and such imaging modalities would definitely help in the understanding of the human diseased conditions. Information gathered from the images can then be added to the present medical database to make it more comprehensive and thus permit the physician to make a more specific diagnosis, prognosis and possibly the appropriate therapy. The basic challenge in this direction is finding the right mathematical frameworks which appropriately describe the processes involved.

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