Chapter 5

On Some Zweier I-Convergent Sequence Spaces Defined by a Modulus Function

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"Good mathematicians see analogies between theorems or theories, the very best ones see analogies between analogies"- Banach.

5.1 Introduction

Ruckle[62-64] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle[62] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[62-64] proved that, for any modulus f,

$$X(f) \subset \ell_1 \text{ and } X(f)^{\alpha} = \ell_{\infty}$$

where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See[62]).}$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in[16]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling[14-15], Köthe[50], Kolk[51-52] and Ruckle[29-31].

In this chapter we introduce the following class of sequence spaces.

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \text{there is } L \in \mathbb{C} \text{ such that}$$
$$\text{for } \varepsilon > 0, \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \text{for a given } \varepsilon > 0, \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M\} \in I, \text{ for each fixed M>0}\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty(f) \cap \mathcal{Z}_0^I(f).$$

5.2 Main Results

Theorem 5.2.1. For any modulus function f, the classes of sequences $\mathcal{Z}^I(f), \mathcal{Z}_0^I(f), m_{\mathcal{Z}}^I(f)$ and $m_{\mathcal{Z}_0}^I(f)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(f)$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in \mathcal{Z}^I(f)$ and let α, β be scalars. Then

$$I - \lim f(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C} \quad ;$$
$$I - \lim f(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C} \quad ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : f(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \qquad [5.1]$$

$$A_2 = \{k \in \mathbb{N} : f(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \qquad [5.2]$$

Since f is a modulus function, we have

$$f(|(\alpha x_k+\beta y_k)-(\alpha L_1+\beta L_2)|)\le f(|\alpha||x_k-L_1|)+f(|\beta||y_k-L_2|)$$
$$\le f(|x_k-L_1|)+f(|y_k-L_2|)$$

Now, by [5.1] and [5.2], $\{k\in$ N: $f(|(\alpha x_k+\beta y_k)-(\alpha L_1+\beta L_2)|)>\epsilon\}$ $\subset A_1\cup A_2$. Therefore $(\alpha x_k+\beta y_k)\in\mathcal{Z}^I(f)$. Hence $\mathcal{Z}^I(f)$ is a linear space.

We state the following result without proof in view of Theorem 5.2.1.

Theorem 5.2.2. The spaces $m^I_{\mathcal{Z}}(f)$ and $m^I_{\mathcal{Z}_0}(f)$ are normed linear spaces, normed by

$$||x_k||_*=\sup_k f(|x_k|). \qquad [5.3]$$

Theorem 5.2.3. A sequence $x=(x_k)\in m^I_{\mathcal{Z}}(f)$ I-converges if and only if for every $\epsilon>0$ there exists $N_\epsilon\in\mathbb{N}$ such that

$$\{k\in\mathbb{N}: f(|x_k-x_{N_\epsilon}|)<\epsilon\}\in m^I_{\mathcal{Z}}(f). \qquad [5.4]$$

Proof. Suppose that $L=I-\lim x$. Then

$$B_\epsilon=\{k\in\mathbb{N}:|x_k-L|<\frac{\epsilon}{2}\}\in m^I_{\mathcal{Z}}(f).\ \text{For all } \epsilon>0.$$

Fix an $N_\epsilon\in B_\epsilon$. Then we have

$$|x_{N_\epsilon}-x_k|\le|x_{N_\epsilon}-L|+|L-x_k|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

which holds for all $k\in B_\epsilon$. Hence $\{k\in\mathbb{N}: f(|x_k-x_{N_\epsilon}|)<\epsilon\}\in m^I_{\mathcal{Z}}(f)$.

Conversely, suppose that $\{k\in\mathbb{N}: f(|x_k-x_{N_\epsilon}|)<\epsilon\}\in m^I_{\mathcal{Z}}(f)$. That is $\{k\in\mathbb{N}: (|x_k-x_{N_\epsilon}|)<\epsilon\}\in m^I_{\mathcal{Z}}(f)$ for all $\epsilon>0$. Then the set

$$C_\epsilon=\{k\in\mathbb{N}: x_k\in[x_{N_\epsilon}-\epsilon, x_{N_\epsilon}+\epsilon]\}\in m^I_{\mathcal{Z}}(f)\ \text{ for all } \epsilon>0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}(f)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(f)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(f)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I_{\mathcal{Z}}(f)$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I_{\mathcal{Z}}(f)$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f(\xi) = I - \lim f(x)$, that is $L = I - \lim f(x)$.

Theorem 5.2.4. Let f and g be modulus functions that satisfy the $\triangle_2$-condition. If X is any of the spaces $\mathcal{Z}^I$, $\mathcal{Z}^I_0, m^I_{\mathcal{Z}}$ and $m^I_{\mathcal{Z}_0}$ etc, then the following assertions hold.
(a) $X(g) \subseteq X(f.g)$,
(b) $X(f) \cap X(g) \subseteq X(f + g)$.

Proof. (a) Let $(x_k) \in \mathcal{Z}^I_0(g)$. Then

$$I - \lim_k g(|x_k|) = 0. \qquad [5.5]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t < \delta$. Write $y_k = g(|x_k|)$ and consider $\lim_k f(y_k) = \lim_k f(y_k)_{y_k<\delta} + \lim_k f(y_k)_{y_k>\delta}$. We have

$$\lim_k f(y_k) \leq f(2) \lim_k (y_k) \qquad [5.6]$$

For $y_k > \delta$, we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since f is non-decreasing, it follows that

$$f(y_k) < f(1 + \frac{y_k}{\delta}) < \frac{1}{2}f(2) + \frac{1}{2}f(\frac{2y_k}{\delta})$$

Since f satisfies the $\triangle_2$-condition, we have

$$f(y_k) < \frac{1}{2}K\frac{y_k}{\delta}f(2) + \frac{1}{2}K\frac{y_k}{\delta}f(2) = K\frac{y_k}{\delta}f(2)$$

Hence

$$\lim_k f(y_k) \leq max(1, K)\delta^{-1}f(2) \lim_k(y_k). \qquad [5.7]$$

From [5.5], [5.6] and [5.7] we have $(x_k) \in \mathcal{Z}_0^I(f.g)$.

Thus $\mathcal{Z}_0^I(g) \subseteq \mathcal{Z}_0^I(f.g)$. The other cases can be proved similarly.

(b) Let $(x_k) \in \mathcal{Z}_0^I(f) \cap \mathcal{Z}_0^I(g)$. Then

$$I - \lim_k f(|x_k|) = 0 \; and \; I - \lim_k g(|x_k|) = 0$$

The rest of the proof follows from the following equality

$$\lim_k(f + g)(|x_k|) = \lim_k f(|x_k|) + \lim_k g(|x_k|).$$

Corollary 5.2.5. $X \subseteq X(f)$ for X =$\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 5.2.6. The spaces $\mathcal{Z}_0^I(f)$ and $m_{\mathcal{Z}_0}^I(f)$ are solid and monotone.

Proof. We shall prove the result for $\mathcal{Z}_0^I(f)$. Let $(x_k) \in \mathcal{Z}_0^I(f)$. Then

$$I - \lim_k f(|x_k|) = 0. \qquad [5.8]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [5.8] and the following inequality

$$f(|\alpha_k x_k|) \leq |\alpha_k| f(|x_k|) \leq f(|x_k|) \; for \; all \; k \in \mathbb{N}.$$

That the space $\mathcal{Z}_0^I(f)$ is monotone follows from the Lemma 5.1.1. For $m_{\mathcal{Z}_0}^I(f)$ the result can be proved similarly.

Theorem 5.2.7. The spaces $\mathcal{Z}^I(f)$ and $m_{\mathcal{Z}}^I(f)$ are neither solid nor monotone in general .

Proof. Here we give a counter example. Let $I = I_\delta$ and $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows.

Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y_k) = \begin{cases} (x_k), & \text{if k is even,} \\ 0, & otherwise. \end{cases}$$

Consider the sequence (x_k) defined by $(x_k) = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{Z}^I(f)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(f)$. Thus $\mathcal{Z}^I(f)$ is not monotone. Hence $\mathcal{Z}^I(f)$ is not solid.

Theorem 5.2.8. The spaces $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I(f)$ is a sequence algebra. Let $(x_k), (y_k) \in \mathcal{Z}_0^I(f)$. Then

$$I - \lim f(|x_k|) = 0$$

and

$$I - \lim f(|y_k|) = 0$$

Then we have

$$I - \lim f(|(x_k.y_k)|) = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I(f)$ is a sequence algebra. For the space $\mathcal{Z}_0^I(f)$, the result can be proved similarly.

Theorem 5.2.9. The spaces $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \text{ and } y_k = k \text{ for all } k \in \mathbb{N}$$

Then $(x_k) \in \mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$, but $(y_k) \notin \mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$. Hence the spaces $\mathcal{Z}_0^I(f)$ and $\mathcal{Z}_0^I(f)$ are not convergence free.

Theorem 5.2.10. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $f(x) = x$ for all $x \in [0, \infty)$. If

$$x_k = \begin{cases} 1, & \text{for } k \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.22 $(x_k) \in \mathcal{Z}_0^I(f) \subset \mathcal{Z}^I(f)$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin \mathcal{Z}^I(f)$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I(f)$. Hence $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are not symmetric.

Theorem 5.2.11. Let f be a modulus function. Then $\mathcal{Z}_0^I(f) \subset \mathcal{Z}^I(f) \subset \mathcal{Z}_\infty^I(f)$.

Proof. Let $(x_k) \in \mathcal{Z}^I(f)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f(|x_k - L|) = 0$$

We have $f(|x_k|) \leq \frac{1}{2}f(|x_k - L|) + f\frac{1}{2}(|L|)$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}^I_\infty(f)$. The inclusion $\mathcal{Z}^I_0(f) \subset \mathcal{Z}^I(f)$ is obvious.

Theorem 5.2.12. The function $\hbar : m^I_{\mathcal{Z}}(f) \to \mathbb{R}$ is the Lipschitz function, where $m^I_{\mathcal{Z}}(f) = \mathcal{Z}^I_\infty(f) \cap \mathcal{Z}^I(f)$, and hence uniformly continuous.

Proof. Let $x, y \in m^I_{\mathcal{Z}}(f)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < ||x - y||_*\} \in m^I_{\mathcal{Z}}(f),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < ||x - y||_*\} \in m^I_{\mathcal{Z}}(f).$$

Hence also $B = B_x \cap B_y \in m^I_{\mathcal{Z}}(f)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m^I_{\mathcal{Z}_0}(f)$ the result can be proved similarly.

Theorem 5.2.13. If $x, y \in m^I_{\mathcal{Z}}(f)$, then $(x.y) \in m^I_{\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I_{\mathcal{Z}}(f),$$

$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I_{\mathcal{Z}}(f).$$

Now,

$$|x_k y_k - \hbar(x)\hbar(y)| = |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)|$$
$$\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)| \qquad [5.9]$$

As $m^I_{\mathcal{Z}}(f) \subseteq \mathcal{Z}^I_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn [5.9] we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m^I(f)$. Hence $(x.y) \in m^I_{\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m^I_{\mathcal{Z}_0}(f)$ the result can be proved similarly.