Chapter 2

The Study of Real Data Using ad hoc Stochastic Models

Section 2.1

Maximum Likelihood Estimation of the Parameters of a Stochastic Differential System Modeling the Returns of the Index of Some Classes of Hedge Funds

[Description] *We test the ability of a stochastic differential model of forecasting the returns of a long-short equity hedge fund index and of a market index, that is of the HFRI-Equity index and of the S&P 500 index respectively. The model is based on the assumptions that the value of the variation of the log-return of the hedge fund index (HFRI Equity) is proportional up to an additive stochastic error to the value of the variation of the log-return of a market index (S&P 500) and that the log-return of the market index can be satisfactorily modeled using the Heston stochastic volatility model. The model is calibrated on observed data using a method based on filtering and maximum likelihood. The data analyzed (i.e. HFRI-Equity and S&P 500 indices) go from January 1990 to June 2007, and are monthly data. The values of the HFRI-Equity and S&P 500 indices forecast by the calibrated models are compared to the values of the indices observed. The result of the comparison is satisfactory.*

[Paper] *Capelli P., Mariani F., Recchioni M.C., Spinelli F., Zirilli F. (2010). Determining a stable relationship between hedge fund index HFRI-Equity and S&P 500 behaviour, using filtering and maximum likelihood, Inverse Problems in Science and Engineering 18, 83-109.*

[Website] *http://www.econ.univpm.it/recchioni/finance/w8*

2.1.1 Outline of the Presentation

- The "long short equity" hedge funds
- A stochastic volatility model for the index of the "long short equity" hedge funds based on the Heston stochastic volatility model
- The forecasting and estimation problems
- Integral representation formula for the solution of the filtering problem
- Integral representation formulae for the forecasted values of the state variables
- The calibration problem
- Numerical experiments with synthetic and real data
- References

2.1.2 "Long/Short Equity" Hedge Funds

The "hedge funds" are "funds" having a "speculative" management.

The regulation of these funds is elastic and this fact gives to the manager a large set of choices. The funds are classified on the basis of the management style and on the market on which they act.

They can be classified in four macro-classes: long/short equity, event driven, relative value, global macro.

The manager of a LONG/SHORT EQUITY fund pursues the goal of constructing a stock portfolio whose return (yield) is independent of the market behaviour, and depends only on the manager ability in stock picking.

The manager buys (long position) the stocks that, in his feeling, are underestimated by the market and sells short (short position) those stocks that he believes are overestimated.

2.1.3 A Single Factor Model for the Index of "Long/Short Equity" Hedge Funds

In the work of Pillonel P., Solanet L.: Predictability in hedge fund index returns and its application in fund of hedge funds style allocation, Master's Thesis in Banking and Finance at Universit?de Lausanne, Hautes Etudes Commerciales (HEC), (2006), the authors show using time series analysis that the return of the index of the long/short equity hedge funds can be explained using the log-return of the $S\&P500$ index by a relation of the type:

$$z_t = a + bx_{t-1} + e_t, \quad t = 1, 2, \ldots$$

- z_t is the hedge fund index return at time t
- x_{t-1} is the S&P500 return at time $t-1$
- e_t is the error term at time t
- a, b are suitable constants.

2.1.4 From a Discrete Time Model to a Continuous Time Model

We propose a reasonable translation in the continuous time setting of the findings of the time series analysis presented by Pillonel and Solanet 2006 that is we consider the following continuous time dynamics for x_t, z_t:

$$z_t = a + bx_{t-1} + e_t, \quad t = 1, 2, \dots \text{ (discrete time)}$$

$$dz_t = \beta dx_t + \gamma\, dW_t, \quad t > 0, \text{ (continuous time)}$$

where β and γ are suitable constants and W_t, $t > 0$, is a standard Wiener process such that $W_0 = 0$ and dW_t is its stochastic differential. Moreover we assume that the dynamics of x_t, $t > 0$, (the $S\&P500$ index) is described by the Heston stochastic volatility model (Heston 1993).

The Heston stochastic volatility model for (x_t, v_t), $t > 0$, coupled with the previous equation for dz_t is the model for the return of the long/short equity hedge funds index that we propose.

2.1.5 The Stochastic Model for "Long/Short Equity" Hedge Funds

Let t be a real variable that denotes time, we consider the stochastic process (x_t, v_t, z_t), $t > 0$, solution of the system of stochastic differential equations:

$$dx_t = (\hat{\mu} - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t^1, t > 0,$$

$$dz_t = \beta(\hat{\mu} - \frac{1}{2}v_t)dt + \sqrt{v_t}\left(\beta dW_t^1 + \gamma\, dW_t^3\right), t > 0,$$

$$dv_t = \chi(\theta - v_t)dt + \varepsilon\sqrt{v_t}dW_t^2, t > 0,$$

where W_t^1, W_t^2, W_t^3 are standard Wiener processes and dW_t^1, dW_t^2, dW_t^3 are their stochastic differentials satisfying:

$$< dW_t^1 dW_t^2 >= \rho_{1,2} dt, \quad < dW_t^1 dW_t^3 >= \rho_{1,3} dt,$$

$$< dW_t^2 dW_t^3 >= \rho_{2,3} dt, t > 0,$$

where $< \cdot >$ denotes the expected value of $\cdot$, $\rho_{1,2}$, $\rho_{1,3}$, $\rho_{2,3} \in [-1, 1]$ are the correlation coefficients and the quantities $\hat{\mu}$, χ, ε, θ, β, γ, are real constants. The stochastic differential equations must be equipped with the initial conditions: $x_0 = \tilde{x}_0$, $z_0 = \tilde{z}_0$, $v_0 = \tilde{v}_0$.

2.1.6 Forecasting and Estimation Problems

DATA

- the observation times $0 = t_0 < t_1 < t_2 < \ldots < t_n < +\infty$;
- the log-return x_t of the stock index: $\tilde{x}_i = x(t_i)$, $i = 0, 1, 2, \ldots, n$;
- the log-return z_t of the index of the long/short equity hedge funds: $\tilde{z}_i = z(t_i)$, $i = 0, 1, 2, \ldots, n$;

We want to use the data available to solve the following problems:

1. Filtering Problem: given the values of the model parameters $\underline{\Theta}= (\hat{\mu}, \chi, \varepsilon, \theta, \beta, \gamma, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}, \tilde{v}_0)^T$ find a forecast of the current variance level (forecasting problem).

2. Estimation Problem: find an estimate of the vector $\underline{\Theta}$.

2.1.7 Filtering Problem

Let us assume that the vector $\underline{\Theta}$ and the filtration $\mathcal{F}_t = \{\, (\tilde{x}_i, \tilde{z}_i) \,|\, t_i \leq t\}$ are given.

The filtering problem consists in finding the joint probability density function $p(x, z, v, t \,|\, \mathcal{F}_t, \underline{\Theta})$ of x_t, z_t, v_t, $t > 0$, conditioned to the observations contained in $\mathcal{F}_t$ for $t > 0$.

The forecasted values $\hat{x}_{t|\underline{\Theta}}$, $\hat{z}_{t|\underline{\Theta}}$, $\hat{v}_{t|\underline{\Theta}}$ of (x_t, z_t, v_t), $t > 0$ can be found as:

$$\hat{x}_{t|\underline{\Theta}}=\mathbb{E}(x_t|\mathcal{F}_t,\underline{\Theta})=\int_0^{+\infty} dv\int_{-\infty}^{+\infty} dz\int_{-\infty}^{+\infty} vdxp(x,z,v,t|\mathcal{F}_t,\underline{\Theta}),t>0,$$

$$\hat{z}_{t|\underline{\Theta}}=\mathbb{E}(z_t|\mathcal{F}_t,\underline{\Theta})=\int_0^{+\infty} dv\int_{-\infty}^{+\infty} dz\int_{-\infty}^{+\infty} dxzp(x,z,v,t|\mathcal{F}_t,\underline{\Theta}),t> v0,$$

$$\hat{v}_{t|\underline{\Theta}}=\mathbb{E}(v_t|\mathcal{F}_t,\underline{\Theta})=\int_0^{+\infty} dv\int_{-\infty}^{+\infty} dz\int_{-\infty}^{+\infty} dxvp(x,z,v,t|\mathcal{F}_t,\underline{\Theta}),t>0.$$

2.1.8 Solution of the Filtering Problem

The joint probability density function $p(x, z, v, t|\mathcal{F}_t, \underline{\Theta})$, $(x, z, v) \in R \times R \times R^+$, $t > 0$, can be obtained starting from the transition probability density function $p_f(x, z, v, t, x', z', v', t'|\underline{\Theta})$, $t, t' > 0$,

$t - t' > 0$, that is from the fundamental solution of the Fokker Planck equation associated to the stochastic differential system (1), (2), (3) as follows: for $i = 0, 1, ..., n$, $(x, z, v) \in R \times R \times R^{+}$:

$$p(x, z, v, t|\mathcal{F}_{t_i}, \underline{\Theta}) = \int_0^{+\infty} p_f(x, z, v, t, \tilde{x}_i, \tilde{z}_i, v', t_i|\underline{\Theta}) f_i(v'; \underline{\Theta}) dv', \quad t_i < t < t_{i+1},$$

where:

$$f_0(v; \underline{\Theta}) = \delta(v - \tilde{v}_0), \quad v \in R^{+},$$

and for $i = 1, 2, ..., n$:

$$f_i(v; \underline{\Theta}) = \frac{p(\tilde{x}_i, \tilde{z}_i, v, t_i^-|\mathcal{F}_{t_{i-1}}, \underline{\Theta})}{\int_0^{+\infty} p(\tilde{x}_i, \tilde{z}_i, v', t_i^-|\mathcal{F}_{t_{i-1}}, \underline{\Theta}) dv'}, \; v \in R^{+}.$$

(see Fatone et al. 2007 and Mariani et al. 2008 for further details)

2.1.9 Integral Representation Formula for the Fundamental Solution

Proceeding as in Lipton, 2001, pag. 605 it is easy to derive the following representation formula for the fundamental solution p_f :

$$
\begin{aligned}
&p_f(x,z,v,t,x',z',v',t'|\underline{\Theta}) = \\
&\quad \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dk e^{\iota k(x-x'-\hat{\mu}\tau)} \int_{-\infty}^{+\infty} d\xi e^{\iota\xi(z-z'-\hat{\mu}\beta\tau)} \cdot \\
&\quad \left\{ e^{\frac{-2}{\varepsilon^2}\chi\theta\mu(k,\xi)\tau} e^{-\frac{2s_\delta(k,\xi,\tau)}{\varepsilon^2 s_\gamma(k,\xi,\tau)}v'} \tilde{M}(k,\xi,\tau)\tilde{A}(k,\xi,\tau)\left(\frac{v}{v'}\right)^{\nu/2} \cdot \right. \\
&\quad \left. e^{-\tilde{M}(k,\xi,\tau)v} I_\nu\left(2\tilde{A}(k,\xi,\tau)\tilde{M}(k,\xi,\tau)\sqrt{v\,v'}\right)\right\} \\
&(x,z,v),\ (x',z',v') \in R\times R\times\mathbb{R}^+,\ t,t'>0,\ \ \tau=t-t'>0,
\end{aligned}
$$

where ι denotes the imaginary unit, $I_\nu(z)$, $z\in\ C$, is the modified Bessel function of positive real order $\nu=\frac{2\chi\theta}{\varepsilon^2}-1$. The other functions appearing in the representation formula are elementary functions of the variables $(k,\xi)\in R\times R$ and $\tau\in\ R^+$.

The functions appearing in the formula for p_f are: for $(k,\,\xi)\in\mathbb{R}\times\mathbb{R},\ \ \tau\in\mathbb{R}^+$,

$$\mu(k,\xi) = -\frac{1}{2}\left\{\chi + \iota\,\varepsilon[k\rho_{1,2} + \xi(\beta\rho_{1,2} + \gamma\rho_{2,3})]\right\},$$

$$\rho(k,\xi) = \frac{1}{2}\Big\{4\mu(k,\xi)^2 + \varepsilon^2\big[k^2 + \xi^2(\beta^2 + \gamma^2 + 2\rho_{1,3}\beta\gamma) +$$

$$2k\xi(\rho_{1,3}\gamma + \beta) - \iota(k + \beta\xi)\big]\Big\}^{1/2},$$

$$s_\gamma(k,\xi,\tau) = 1 - e^{-2\rho(k,\xi)\tau},$$

$$s_\beta(k,\xi,\tau) = \rho(k,\xi)(1 + e^{-2\rho(k,\xi)\tau}) - \mu(k,\xi)s_\gamma(k,\xi,\tau),$$

$$s_\delta(k,\xi,\tau) = \rho(k,\xi)(1 + e^{-2\rho(k,\xi)\tau}) + \mu(k,\xi)s_\gamma(k,\xi,\tau),$$

$$\tilde{M}(k,\xi,\tau) = \frac{2s_\beta(k,\xi,\tau)}{\varepsilon^2 s_\gamma(k,\xi,\tau)},$$

$$A(k,\xi,\tau) = \frac{2\rho(k,\xi)e^{-\rho(k,\xi)\tau}}{s_\beta(k,\xi,\tau)}.$$

2.1.10 Integral Representation Formulae for the Forecasted Values

From the knowledge of the joint probability density function $p(x, z, v, t|\mathcal{F}_t, \underline{\Theta})$, $(x, z, v) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$, $t \geq 0$, we derive the following formulae that are used to forecast the values of the stock index log-return and of the hedge fund index log-return x_t, z_t, $t > 0$, $t \neq t_i$, $i = 1, 2, \ldots, n$, respectively and of the stochastic variance v_t, $t > 0$:

$$\mathbb{E}(x_t|\mathcal{F}_{t_i},\underline{\Theta}) = \tilde{x}_i + \hat{\mu}(t-t_i) + \theta\frac{(1-e^{-\chi(t-t_i)})}{2\chi} -$$
$$\frac{(1-e^{-\chi(t-t_i)})}{2\chi}\int_0^{+\infty} dv\, v\, f_i(v;\underline{\Theta}),$$
$$\mathbb{E}(z_t|\mathcal{F}_{t_i},\underline{\Theta}) = \tilde{z}_i + \beta\,\hat{\mu}(t-t_i) + \theta\frac{\beta(1-e^{-\chi(t-t_i)})}{2\chi} -$$
$$\frac{\beta(1-e^{-\chi(t-t_i)})}{2\chi}\int_0^{+\infty} dv\, v\, f_i(v;\underline{\Theta}),$$
$$\mathbb{E}(v_t|\mathcal{F}_{t_i},\underline{\Theta}) = \theta(1-e^{-\chi(t-t_i)}) + e^{-\chi(t-t_i)}\int_0^{+\infty} dv\, v f_i(v;\underline{\Theta}),$$
$$t_i \le t < t_{i+1},\ i = 0,1,\ldots,n\,.$$

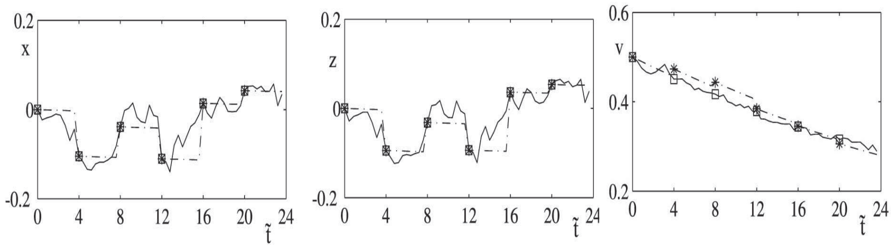
Note that we have reduced the computation of the forecasted value to "one-dimensional" integrals.

2.1.11 Forecasted Values of x_t, z_t, v_t

Parameter values $\hat{\mu} = 0.026$, $\chi = 5.94$, $\varepsilon = 0.306$, $\theta = 0.01159$, $\rho_{1,2} = -0.576$, $\rho_{1,3} = 0$, $\rho_{2,3} = 0$, $\tilde{v}_0 = 0.5$, $\beta = 1$, $\gamma = 0.1$, $dt = 4/252.5$.

solid line: "true trajectory"

dash-dotted line: "forecasted trajectory"



2.1.12 Estimation Problem

The solution of the estimation problem is the vector $\underline{\Theta} = (\hat{\mu}, \chi, \varepsilon, \theta, \beta, \gamma, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}, \tilde{v}_0)^T$ belonging to the set $\mathcal{M} = \{\underline{\Theta} = (\hat{\mu}, \chi, \varepsilon, \theta, \beta, \gamma, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}, \tilde{v}_0)^T \in \mathbb{R}^{10} \mid \chi > 0, \varepsilon > 0, \theta > 0, \beta \geq 0, \gamma \geq 0, \frac{2\chi\theta}{\varepsilon^2} \geq 1, 1 \geq \rho_{1,2}, \rho_{1,3}, \rho_{2,3} \geq -1, \tilde{v}_0 > 0\}$ that makes most likely the observations $(\tilde{x}_i, \tilde{z}_i)$, at time $t = t_i$, $i = 0, 1, 2, \ldots, n$, that is the vector $\underline{\Theta}$ that solves the following problem:

$$\max_{\underline{\Theta} \in \mathcal{M}} F(\underline{\Theta}).$$

where

$$F(\underline{\Theta}) = \sum_{i=0}^{n-1} \log\left\{\int_0^{+\infty} p(x_{i+1}, z_{i+1}, v', t_{i+1}^- | \mathcal{F}_{t_i}, \underline{\Theta}) dv'\right\}, \underline{\Theta} \in \mathcal{M}.$$

We call $F(\underline{\Theta})$ (log-) likelihood function.

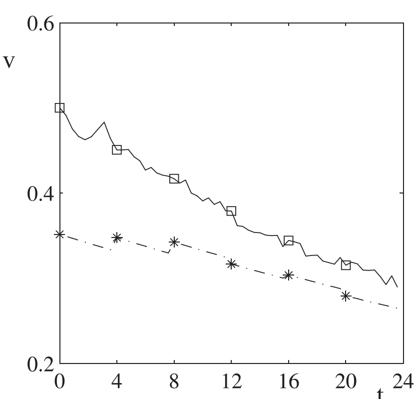
The technique used to solve the previous optimization problem is based on a variable metric steepest ascent method. Beginning from an initial guess $\underline{\Theta}^0$, we update at every iteration the current approximation of the solution with a step in the direction of the gradient of the (log-)likelihood function computed in a suitable metric (see Fatone et al. 2007).

2.1.13 Numerical Results on Synthetic Data

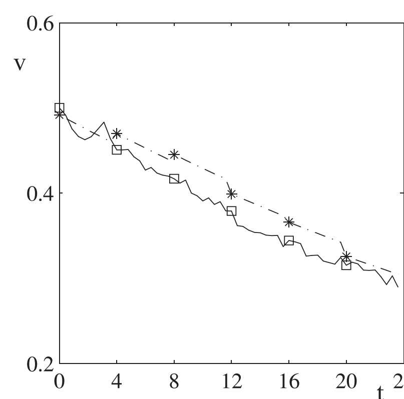
Let us remember the dynamical system

$$\begin{aligned} dx_t &= (\hat{\mu} - \tfrac{1}{2}v_t)dt + \sqrt{v_t}dW_t^1,\ t > 0, \\ dz_t &= \beta(\hat{\mu} - \tfrac{1}{2}v_t)dt + \\ &\qquad \sqrt{v_t}\left(\beta dW_t^1 + \gamma\, dW_t^3\right),\ t > 0, \\ dv_t &= \chi(\theta - v_t)dt + \varepsilon\sqrt{v_t}dW_t^2,\ t > 0. \end{aligned}$$

We choose the following parameters $\underline{\Theta} = (\hat{\mu}, \chi, \varepsilon, \theta, \beta, \gamma, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}, \tilde{v}_0)^T$ $= \underline{\Theta}_1 =$ (0.026, 5.94,0.306,0.01159,1,0.1,-0.576,0,0,0.5$)^T$ and we use six observation $t_i = 4\,i/252.5$, $i = 0, 1, 2, 3, 4, 5$ (see D.S.Bates, The Review of Financial Studies, 19 (2006), 909-965).



k=0, $\underline{\Theta}=\underline{\Theta}^0$

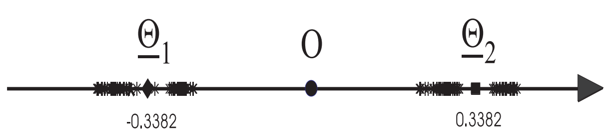


k=19, $\underline{\Theta}=\underline{\Theta}^{19}$

Animation▷

2.1.14 Analysis of a Two Years Time Series

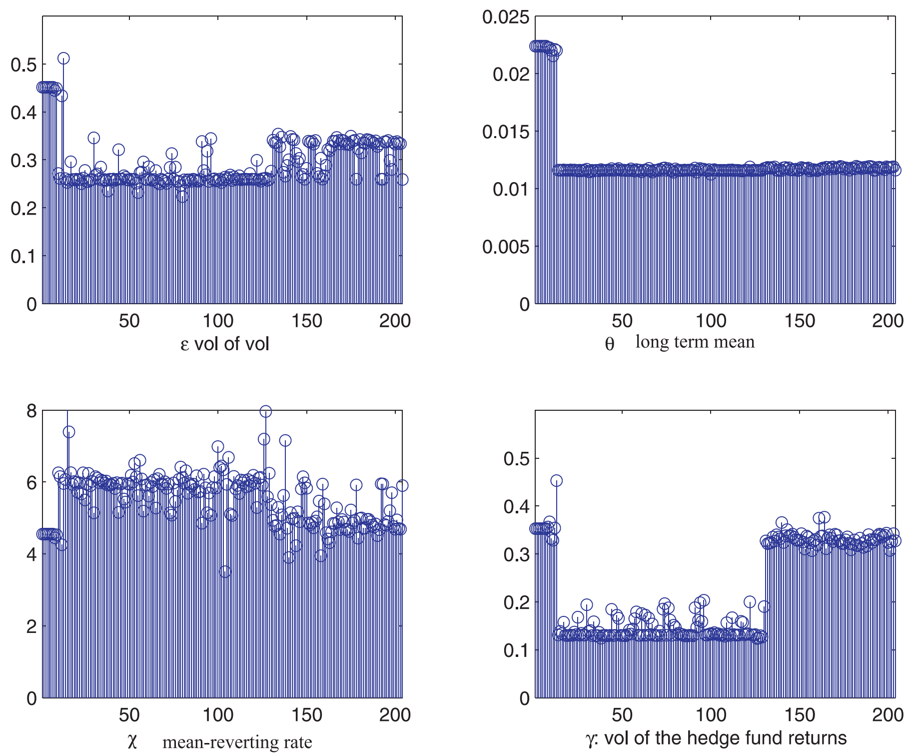
We have generated the data integrating the stochastic differential system for a two years period using the following parameter vectors in the first year $\underline{\Theta} = (\hat{\mu}, \ \chi, \ \varepsilon, \ \theta, \ \beta, \ \gamma, \ \rho_{1,2}, \ \rho_{1,3}, \ \rho_{2,3}, \tilde{v}_0)^T = \underline{\Theta}_1 = (0.026, 5.94, 0.306, 0.01159, 1, 0.1, -0.576, 0, 0, 0.5)^T$ and in the second year we have $\underline{\Theta} = (\hat{\mu}, \ \chi, \ \varepsilon, \ \theta, \beta, \ \gamma, \ \rho_{1,2}, \ \rho_{1,3}, \ \rho_{2,3}, \ \tilde{v}_0)^T$ = $\underline{\Theta}_2$=$(0.4, 2, 0.01, 0.01, 1, 0.01, 0.5, 0, 0, 0.012)^T$. The series made of 505 observation times corresponding to 1010 observations $(\tilde{x}_i, \tilde{z}_i)$, $i = 0, 1, 2, \ldots, 504$. We solve the optimization problem using a time window made of 8 consecutive observation times. In particular we divide the time interval $[0, \frac{504}{252.5}]$ in 63 consecutive disjoint time windows containing 8 consecutive observation times. the results obtained are summarized in the following figure.



Animation▷

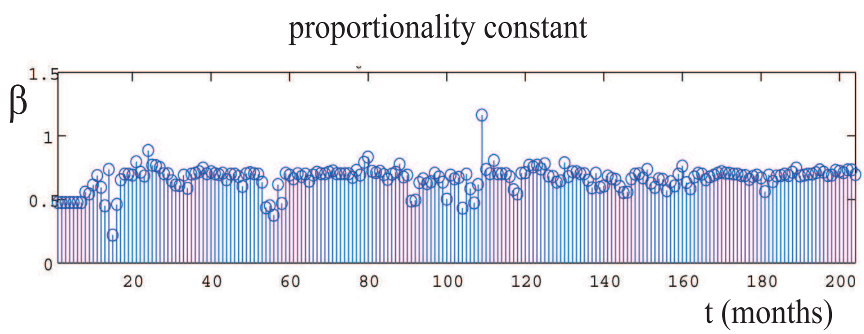
2.1.15 Numerical Results on Real Data (Banca Akros, Milano)

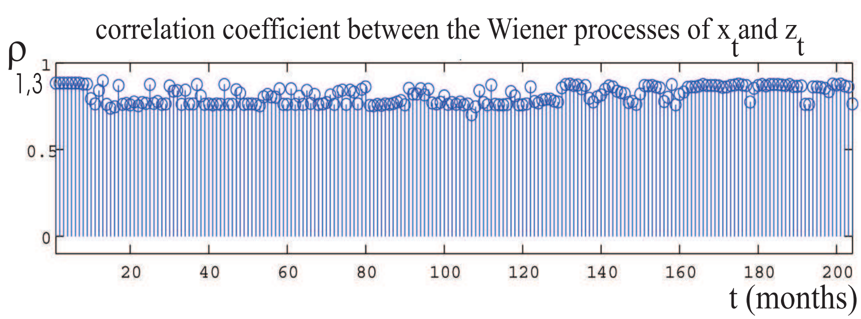
We have 211 observation times corresponding to 211 months from 1/31/1990 to 06/30/2007, that is we have 211 couples $(\tilde{x}_i, \tilde{z}_i)$, $i = 0, 1, \ldots, 210$. We have applied the calibration procedure on a window of nine consecutive observation times $(\tilde{x}_i, \tilde{z}_i)$, $i = 0, 1, 2, \ldots, 8$. We move this window through the data time series discarding the data corresponding to the first observation time of the window and inserting the data corresponding to the next observation time after the window. So that we solve 203 optimization problems and we obtain the reconstruction of the parameters shown below.


0.5
0.4
0.3
0.2
0.1
0
50
100
150
200
ε vol of vol
0.025
0.02
0.015
0.01
0.005
0
50
100
150
200
θ long term mean
8
6
4
2
0
50
100
150
200
χ mean-reverting rate
0.5
0.4
0.3
0.2
0.1
0
50
100
150
200
γ: vol of the hedge fund returns

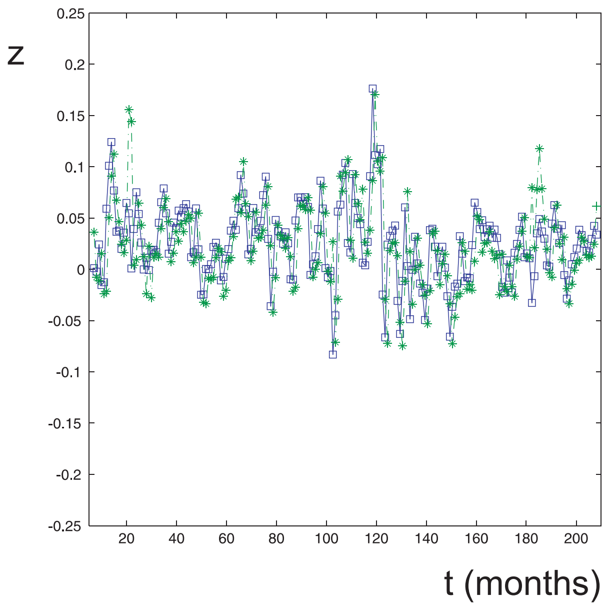
2.1.16 Is the Model for the Return of the Index of the Hedge Funds a Satisfactory Model?

Let us show the reconstruction of the two parameters β, $\rho_{1,3}$. Remember that we have assumed $z_t \approx \beta x_t$ hence we expect that β is constant and that z_t and x_t are positively correlated.





2.1.17 Comparison Between Forecasted Values and Data of the Hedge Fund Index Returns



Mean absolute error on the forecasted values 0.0287

2.1.18 Future Work

- Solve filtering problems that uses the prices of some derivatives on the indices considered as data to make more accurate the calibration procedure.
- Derive semi explicit formulae in the limit case of high frequency data to reduce the computational cost of the solution of the filtering problem.
- Extend the previous work to other kinds of hedge funds suggesting

and analyzing adequate models.

Note that several numerical experiments and digital movies relative to the problem considered here that show the behaviour of the filtering and estimation method proposed can be found at the website:
http://www.econ.univpm.it/recchioni/finance/w5.

A more general reference to the work in mathematical finance of the authors and of their coauthors is the website:
http://www.econ.univpm.it/recchioni/finance.

2.1.19 References

[1] Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2007). Maximum likelihood estimation of the parameters of a system of stochastic differential equations that models the returns of the index of some classes of hedge funds, Journal of Inverse and Ill-Posed Problems 15, 329–362.

[2] Heston S. (1993). A closed form solution for options with volatility with applications to bond and currency options, Review of Financial Studies 6, 327-343.

[3] Lipton A. (2001). Mathematical methods for foreign exchange, World Scientific Pubblishing Co. Pte. Ltd, Singapore.

[4] Mariani F., Pacelli G., Zirilli F. (2008). Maximum Likelihood Estimation of the Heston Stochastic Volatility Model Using Asset and Option Prices: an Application of Nonlinear Filtering Theory, Optimization Letters 2, 177-222.

[5] Pillonel P., Solanet L. (2006). Predictability in hedge fund index returns and its application in fund of hedge funds'style allocation, Master's Thesis in Banking and Finance at Universit?de Lausanne, Hautes Etudes Commerciales (HEC).

Section 2.2

Calibration of a Stochastic Model of Spiky Prices: An Application to Electric Power Prices

[Description] *We use filtering and maximum likelihood methods to solve a calibration problem for a stochastic dynamical system used to model spiky asset prices. The data used in the calibration problem are the observations at discrete times of the asset price. The model considered describes spiky asset prices through a stochastic process that can be represented as the product of two independent Markov processes: the spike process and the process that represents the asset prices in absence of spikes. A Markov chain is used to regulate the transitions between presence and absence of spikes. Given the calibrated model we develop a sort of tracking procedure able to forecast the forward asset prices. Numerical examples using synthetic and real data of the solution of the calibration problem and of the performance of the tracking procedure are presented. The real data studied are electric power prices data taken from the U.K. electricity market. The forward prices forecast with the tracking procedure and the observed forward prices are compared to evaluate the quality of the model and of the forecasting procedure.*

[Paper] *Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2012). The analysis of real data using a stochastic dynamical system able to model spiky prices, Journal of Mathematical Finance 2, 1-12.*

[Website] *http://www.econ.univpm.it/recchioni/finance/w10*

2.2.1 The Calibration Problem

We want to estimate the parameters of a stochastic dynamical system used to model spiky prices starting from observed data.

We use as data the observations at discrete times of asset prices.

The solution of the calibration problem proposed makes use of:

- nonlinear filtering techniques (see Mariani et al. 2008),
- maximum likelihood method (see Fatone et al. 2007, 2012, 2013).

We focus on:

1. The stochastic dynamical system used to model spiky asset prices;
2. The formulation and the solution of the calibration problem for the spiky asset prices model;
3. The computational efficiency of the solution method of the calibration problem.

The model considered can be used in realistic situations and the analysis of time series of real data has been done successfully.

The real data studied are *electric power prices data taken from the U.K. electricity market.* These data are "spiky" asset prices.

The numerical results obtained are satisfactory (see Fatone et al. 2012).

2.2.2 The Model for Spiky Asset Prices

Following Kholodnyi 2004, 2008 we model spiky asset prices as a non-Markovian stochastic process that can be represented as a product of two independent Markov processes:

- the spike process: this process is responsible for modeling spikes in asset prices. It is either equal to the multiplicative amplitude of the spike during the spike periods or to one during the regular periods, i.e. the periods between two spikes.
- the process that describes the asset prices in absence of spikes: this process is responsible for modeling prices in absence of spikes. In our work this process is a diffusion process.

Finally, the presence or absence of spikes depends on a two-state Markov process in continuous time that determines whether asset prices are in the *spike state* (i.e. during a spike period) on in the *regular state* (i.e. between two spike periods).

Let M_t be this two-state Markov process depending on continuous time $t \geq 0$, and let

$$P(T,t) = \begin{pmatrix} P_{ss}(T,t) & P_{sr}(T,t) \\ P_{rs}(T,t) & P_{rr}(T,t) \end{pmatrix}, \quad 0 < t \leq T,$$

be its 2×2 transition probability matrix.

1. $P_{ss}(T,t)$ and $P_{rs}(T,t)$ denote the transition probabilities of going from the spike state at time t respectively to the spike or to the regular state at time T;

2. $P_{sr}(T,t)$ and $P_{rr}(T,t)$ denote the transition probabilities of going from the regular state at time t respectively to the spike or to the regular state at time T.

Note that the Chapman-Kolmogorov equation for the two-state Markov process M_t, $t \geq 0$, can be written as follows:

$$P(T,t) = P(T,\tau)P(\tau,t), \quad 0 < t \leq \tau \leq T,$$

together with the condition that $P(T,t)$ is the 2×2 identity matrix when $t = T$.

In the important case of a *time-homogeneous* Markov process M_t, $t \geq 0$, the transition probability matrix $P(T,t)$, $0 < t \leq T$ is, in fact, a function of the difference $T - t$, and it can be written as follows:

$$P(T,t) = \begin{pmatrix} \frac{b+ae^{-(T-t)(a+b)}}{a+b} & \frac{b-be^{-(T-t)(a+b)}}{a+b} \\ \frac{a-ae^{-(T-t)(a+b)}}{a+b} & \frac{a+be^{-(T-t)(a+b)}}{a+b} \end{pmatrix}, 0 < t \leq T, \tag{1}$$

where the quantities a and b are real non negative parameters able to control the duration and the frequency of the spike periods (i.e. the expected lifetime of spikes and the expected time between spikes), that is a and b satisfy the following conditions:

$$a \geq 0, b \geq 0.$$

In our model for spiky prices we always assume the two-state Markov process M_t, $t \geq 0$, to be time-homogeneous with the transition probability matrix $P(T,t)$ given by (1).

Let $p_r(t)$, $p_s(t)$ and $p_r(T)$, $p_s(T)$ be respectively the probabilities of being in the regular state and in the spike state at time t and at time T,

$0 < t \leq T$, we have:

$$\begin{pmatrix} p_s(T) \\ p_r(T) \end{pmatrix} = P(T,t) \begin{pmatrix} p_s(t) \\ p_r(t) \end{pmatrix}, \quad 0 < t \leq T.$$

We model the process that describes the asset prices in absence of spikes through a diffusion process defined by the following stochastic differential equation and initial condition:

$$\begin{cases} d\hat{S}_t = \mu \hat{S}_t dt + \sigma \hat{S}_t dW_t, & t > 0, \\ \hat{S}_0 = \hat{S}^*, \end{cases} \tag{2}$$

where

1. $\hat{S}_t > 0$ denotes the asset prices in absence of spikes at time $t \geq 0$,
2. μ is the drift coefficient,
3. $\sigma > 0$ is the volatility coefficient,
4. W_t is the standard Wiener process, $W_0 = 0$, and dW_t is its stochastic differential,
5. $\hat{S}^* > 0$ is a given initial condition.

Equation (2) defines the asset price dynamics of the celebrated Black Scholes model.

Note that (2) is a Markov process. In Kholodnyi 2004 underlines that other Markov processes different from (2) can be used to model asset price dynamics in absence of spikes.

We define the spike process λ_t, $t \geq 0$, that will be responsible for modeling the amplitude of the spikes in the asset prices as follows:

let ξ_t, $t \geq 0$, be a stochastic process made of independent random variables with given probability density functions $\Sigma(t, \xi),\ \xi > 0,\ t \geq 0$.

We assume that:

1. If the Markov process M_t is in the regular state then the spike process λ_t is equal to one, i.e. $\lambda_t = 1$.
2. If the Markov process M_t transits into the spike state at time τ then the spike process λ_t is equal to a value sampled from the random variable ξ_τ during the entire time that the Markov process M_t remains in the spike state. (We assume that M_t is in the regular state at time $t = 0$ with probability one, so that the spike process λ_t starts with $\lambda_0 = 1$ with probability one).

Note that λ_t, $t \geq 0$, is the magnitude of the multiplicative amplitude of the spikes when the transition to the spike state happens at time $t \geq 0$ and that $\Sigma(t, \xi),\ t \geq 0$, is the probability density function of λ_t, $t \geq 0$, for the spike period that begins at time t.

Let us observe that in the special case of spikes with constant amplitude $\lambda > 1$, the probability density function $\Sigma(t, \lambda') = \Sigma(\lambda')$, $\lambda' > 0$, $t > 0$, is the Dirac delta function $\delta(\lambda - \lambda')$, i.e.:

$$\Sigma(\lambda') = \delta(\lambda - \lambda').$$

Finally we say that the spike process λ_t is in the *spike state* or *regular state* if the Markov process M_t is in the spike state or regular state respectively.

In our model for spiky prices we assume that the spikes have constant amplitude $\lambda \geq 1$.

2.2.3 The Process for Asset Prices with Spikes

Let us denote with $S_t > 0$ the price (eventually) with spikes of the asset at time $t \geq 0$.

Let us assume that the spike process λ_t and the process $\hat{S}_t$ for asset prices in absence of spikes are independent.

We define the process $S_t > 0$, $t \geq 0$, that describes the spiky asset prices, as the product of the spike process λ_t and of the process $\hat{S}_t$ for asset prices in absence of spikes, that is:

$$S_t = \lambda_t \hat{S}_t, \quad t > 0.$$

Note that the process $S_t > 0$, $t \geq 0$, for spiky asset prices is in the *spike state* or in the *regular state* depending from the fact that the spike process λ_t, $t \geq 0$, is in the spike state or in the regular state respectively, or equivalently, depending from the fact that the Markov process M_t, $t \geq 0$, is in the spike state or in the regular state respectively.

Remarks

1. It can be shown (see Kholodnyi 2004, 2008) that, although the process S_t, $t > 0$, is non-Markovian, it can be represented as a Markov process that for any time $t > 0$ can be fully characterized by the values of the processes λ_t and $\hat{S}_t$ at time $t > 0$.

2. Kholodnyi 2004 shows that the process S_t, $t > 0$, can mimic spikes in asset prices, that is S_t, $t > 0$, can exhibit sharp upward price movements shortly followed by equally sharp downward price movements of approximately the same magnitude, so that a spike can form.

In particular, since the *expected times* $\bar{t}_s$ and $\bar{t}_r$ *spent by the process* S_t, $t > 0$, in the *spike state* and in the *regular (i.e.* inter-*spike) state*, respectively, coincide with those associated to the asymptotic probabilities of the Markov process M_t, it can be shown that if M_t is time-homogeneous, we have:

$$\bar{t}_s = \frac{1}{a}, \qquad \bar{t}_r = \frac{1}{b}. \tag{3}$$

3. If $\bar{t}_s$ is small in comparison with the characteristic time of change of the process $\hat{S}_t$, $t > 0$, then we can say that the process S_t, $t > 0$, describes asset prices with spikes. For example, if $\hat{S}_t$, $t > 0$, is the diffusion process of the Black Scholes model then the previous condition can be stated as follows:

$$\sigma^2 \bar{t}_s = \frac{\sigma^2}{a} \ll 1 \quad \text{and} \quad \mu \bar{t}_s = \frac{\mu}{a} \ll 1.$$

That is we can interpret $\bar{t}_s$ as the expected lifetime of a spike, and $\bar{t}_r$ as the expected time between two consecutive spikes. In this way the parameters a and b control the duration and the frequency of the spikes. For example, equations (3) suggest that to model short-lived spikes the parameter a must be chosen to be relatively large, while to model rare spikes the para-meter b must be chosen to be relatively small .

4. The *asymptotic probabilities* of the time-homogeneous Markov process

M_t, $t \geq 0$, of being in the spike and in the regular states are given respectively, by:

$$p_s(\infty) = \pi_s = \frac{b}{a+b}, quad\, p_r(\infty) = \pi_r = \frac{a}{a+b}.$$

5. In the special case of short-lived spikes with constant amplitude, the expected lifetime of a spike $\bar{t}_s$ is relatively short with respect to the expected time between spikes $\bar{t}_r$, that is:

$$\bar{t}_s \ll \bar{t}_r, \quad \text{i.e.} \quad a \gg b.$$

If we define the *characteristic lifetime of a spike t_{ch}* as $t_{ch} = \dfrac{\bar{t}_s}{\bar{t}_r} = \dfrac{b}{a}$, then the asymptotic probabilities π_s and π_r can be represented as follows:

$$\pi_s = t_{ch} + o(t_{ch}), \pi_r = 1 - t_{ch} + o(t_{ch}), \text{when } t_{ch} \to 0,$$

where $o(t_{ch})$ stands for a term of the order higher than t_{ch} when $t_{ch} \to 0$.

The parameters that must be estimated from the data in the calibration problem are:

- the Black-Scholes model parameters: μ, σ,
- the spiky asset prices model parameters: a, b, λ,

that is, the following vector:

$$\underline{\Theta} = (\mu, \sigma, a, b, \lambda)^T,$$

where ^{T} denotes the transpose operator.

The vector $\underline{\Theta}$ is the unknown of the calibration problem.

The following set of constraints must be satisfied by the vectors $\underline{\Theta}$ that describe admissible sets of parameters:

$$\mathcal{M}=\{\underline{\Theta}=(\mu,\sigma,a,b,\lambda)^T\in\mathbb{R}^5|\sigma\geq 0, a\geq 0, b\geq 0, \lambda\geq 1\}.$$

2.2.4 The Calibration Problem (More)

DATA

- the observation times $0=t_0<t_1<t_2<\ldots<t_n<+\infty$;
- the spiky asset prices S_i observed at time t_i,i=0,1,2,. . .,n.

We want to use these data to solve the following problems:

- Calibration Problem: find an estimate of the vector $\underline{\Theta}=(\mu,\sigma,$ $a,b,\lambda)^T$.
- Filtering Problem (Forecasting Problem): given the values of the model parameters $\underline{\Theta}=(\mu,\sigma,a,b,\lambda)^T$ forecast the *forward prices*.

Note that with forward prices we mean prices "in the future", that is future prices associated to the spot prices S_i, observed at time t_i $i=0,1,\ldots,n$. The meaning of future at time t_i is simply $t>t_i$. Let us observe that for each spot price we can forecast a series of forward prices associated to it.

That is the calibration problem consists in estimating the vector $\underline{\Theta}$ from the data given by the observations at time $t=t_i$ of the asset prices containing spikes $S_i = \lambda_{t_i}\hat{S}_i$, for $i = 0, 1, \ldots, n$, i.e. consists in estimating the value of the vector $\underline{\Theta}$ that makes most likely the available observations $\mathcal{F}_t = \{S_i = \lambda_{t_i}\hat{S}_i : t_i \leq t\}$, $t > 0$.

As a byproduct of the solution of this calibration problem we obtain a technique to track the forward prices.

Remark

For simplicity we assume that the transitions from regular state to spike state or viceversa happen in the observation times.

2.2.5 Solution of the Calibration Problem

Let

1. $p(\hat{S}, t|\mathcal{F}_t, \underline{\Theta})$ be the probability density function of the stochastic process $\hat{S}_t$ at time $t > 0$ conditioned to the observations $\mathcal{F}_t$;

2. $p_i(\hat{S}, t|\underline{\Theta}) = p(\hat{S}, t|\mathcal{F}_{t_i}, \underline{\Theta})$ be the probability density function of the stochastic process $\hat{S}_t$ conditioned to the observations made up to time $t = t_i$, $t_i < t \leq t_{i+1}$, $i = 0, 1, \ldots, n$ where we define $t_{n+1} = +\infty$.

In order to measure the likelihood of the vector $\underline{\Theta}$ we introduce a (log-)likelihood function:

$$F(\underline{\Theta}) = \sum_{i=0}^{n-1} \log p_i(\hat{S}_{i+1}, t_{i+1}|\underline{\Theta}), \quad \underline{\Theta} \in \mathcal{M}.$$

The solution of the calibration problem is given by the vector $\underline{\Theta}$ that solves the following optimization problem:

$$\max_{\underline{\Theta}\in\mathcal{M}} F(\underline{\Theta}). \tag{4}$$

This problem is called maximum likelihood problem and is an optimization problem with nonlinear objective function and linear inequality constraints.

In order to solve problem (4), we must evaluate the (log-) likeli-hood function $F(\underline{\Theta})$, i.e. we must evaluate the probability density functions:

$$p_i(\hat{S},t|\underline{\Theta}), \hat{S}\geq 0, t_i < t \leq t_{i+1}, \underline{\Theta}\in\mathcal{M}, fori = 0,1,...,n.$$

The probability density functions p_i, $i = 0,1,...,n-1$, are solutions of the following Fokker-Planck equation associated to the Black-Scholes model: for $i = 0,1,...,n-1$,

$$\begin{cases} \dfrac{\partial p_i}{\partial t} = -\dfrac{1}{2}\sigma^2\hat{S}^2\dfrac{\partial^2 p_i}{\partial \hat{S}^2} - r\hat{S}\dfrac{\partial p_i}{\partial \hat{S}} + rp_i, \hat{S}\geq 0, t_i < t \leq t_{i+1}, \\ p_i(\hat{S},t_i|\underline{\Theta}) = f_i(\hat{S};\underline{\Theta}), \quad \hat{S}\geq 0, \end{cases} \tag{5}$$

where

$$f_0(\hat{S};\underline{\Theta}) = \delta(\hat{S}-\hat{S}^*),$$
$$f_i(\hat{S};\underline{\Theta}) = \frac{\left[p_r(t_i)\delta(\hat{S}-S_i) + p_s(t_i)\delta\left(\hat{S}-\dfrac{S_i}{\lambda}\right)\right] p_{i-1}(\hat{S},t_i|\underline{\Theta})}{p_r(t_i)p_{i-1}(S_i,t_i|\underline{\Theta}) + p_s(t_i)p_{i-1}\left(\dfrac{S_i}{\lambda},t_i|\underline{\Theta}\right)}, \tag{6}$$
$$i = 1,2,...,n,$$

where $p_r(t_i)$ and $p_s(t_i)$ are respectively the probabilities of the time-homogeneous Markov process M_t, $t \geq 0$, of being in the regular and in the spike state at time $t = t_i$.

Remark

The conditioned probability density functions p_i, $i = 0, 1, ..., n-1$, solutions of the initial value problems (5), (6) for the Fokker-Planck equation, can be written as an integral with respect to the state variable of the product of the fundamental solution of the Fokker-Planck equation associated to the Black- Scholes model with the initial conditions (6).

2.2.6 The Filtering Problem

Let us assume that the vector $\underline{\Theta}$ and $\mathcal{F}_t = \{S_i = \lambda \hat{S}_i : t_i \leq t\}$, $t > 0$ are given.

From the knowledge of the values of the model parameters $\underline{\Theta} = (\mu, \sigma, a, b, \lambda)^T$ we can forecast the *power forward prices* as follows:

$$\mathbb{E}(S_{\tau_i}^{Forw}) = \mathbb{E}(\lambda_{\tau_i})\mathbb{E}(\hat{S}_{\tau_i}) = (1 \cdot p_r(\tau_i) + \lambda \cdot p_s(\tau_i))\mathbb{E}(\hat{S}_{t_i})e^{\mu(\tau_i - t_i)},$$
$$\tau_i = t_i + \Delta t.$$

where $\mathbb{E}(\cdot)$ denotes the mean value of $\cdot$.

Note that we use the following approximation: $\mathbb{E}(\hat{S}_{t_i}) = \frac{1}{10}\sum_{k=0}^{9} \hat{S}_{i-k}$ since the average in time of the observations gives a better approximation of the "spatial" average than the single measure $\hat{S}_i$.

The optimization algorithm used to solve the maximum likelihood problem.

The technique used to solve the maximum likelihood problem is based on a variable metric steepest ascent method.

Beginning from an initial guess $\underline{\Theta}^0$, we update at every iteration the current approximation of the solution of the optimization problem with a step in the direction of the gradient of the (log-)likelihood function computed in a suitable variable metric to take care of the constraints.

Note that the initial guess $\underline{\Theta}^0 \in \mathcal{M}$ is built with some elementary ad hoc steps.

Let us fix a tolerance value $\delta > 0$ and a maximum number of iterations $iter > 0$, we denote with $\underline{\Theta}^*$ the (numerically computed) maximizer of the (log-)likelihood function.

1. Set $k = 0$ and initialize $\underline{\Theta} = \tilde{\underline{\Theta}}^0$;
2. Evaluate $F(\underline{\Theta}^k)$, if $k > 0$ and if $|F(\underline{\Theta}^k) - F(\underline{\Theta}^{k-1})| < \delta$, where $|\cdot|$ denotes the absolute value of $\cdot$, go to item 7;
3. Evaluate the gradient of the (log-)likelihood function: $\nabla F(\underline{\Theta}^k) = \left(\frac{\partial F}{\partial \mu}, \frac{\partial F}{\partial \sigma}, \frac{\partial F}{\partial a}, \frac{\partial F}{\partial b}, \frac{\partial F}{\partial \lambda}\right)^T (\underline{\Theta}^k)$, if $\|\nabla F(\underline{\Theta}^k)\| < \delta$ where $||\cdot||$ denotes the Euclidean norm of the vector $\cdot$, go to item 7;
4. Perform the steepest ascent step, evaluating $\underline{\Theta}^{k+1} = \underline{\Theta}^k + \eta_k \nabla F(\underline{\Theta}^k)$, where η_k is a positive real number representing the length of the step done in the direction of $\nabla F(\underline{\Theta}^k)$. The choice of η_k involves the use of “variable metrics”;

5. If $||\underline{\Theta}^{k+1} - \underline{\Theta}^k|| < \delta$, go to item 7;

6. Set $k = k + 1$, if $k < iter$ go to item 2;

7. Set $\underline{\Theta}^* = \underline{\Theta}^k$ and stop.

2.2.7 Some Numerical Results on Real Data

The real spiky data studied are *electric power prices data taken from the U.K. electricity market*. These data are "spiky" asset prices.

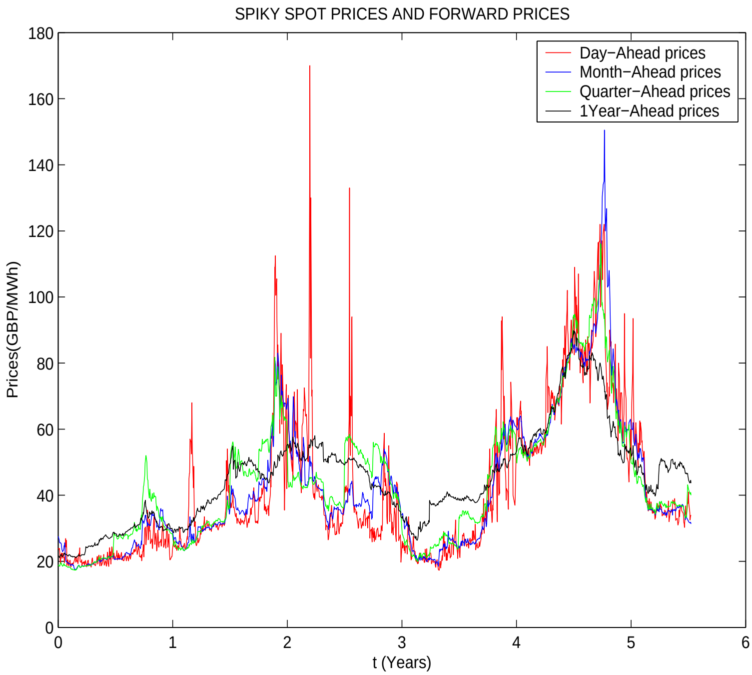
DATA

- the observation times $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1395 < +\infty$ (days)(more than 5 years of daily observations: from 01/05/2004 to 07/10/2009);
- the spiky asset prices S_i at time t_i, where S_i=spot daily electric power price (GBP/MWh), namely Day-Ahead price, $i = 0, 1, 2, \ldots, n$.

For each spot price there is a series of forward prices associated to it for a variety of delivery periods. These include:

- forward price 1 month deep in the future (Month-Ahead price);
- forward price 3 months deep in the future (Quarther-Ahead price);
- forward price 4 months deep in the future (Season-Ahead price);
- forward price 1 year deep in the future (1 Year-Ahead price);

These forward prices are observed each day t_i and are associated to the spot price S_i, $i = 0, 1, 2, \ldots, 1395$.


SPIKY SPOT PRICES AND FORWARD PRICES
Day−Ahead prices
Month−Ahead prices
Quarter−Ahead prices
1Year−Ahead prices
Prices(GBP/MWh)
t (Years)
0
20
40
60
80
100
120
140
160
180
0
1
2
3
4
5
6

2.2.8 Is the Model for Spiky Prices a Satisfactory Model?

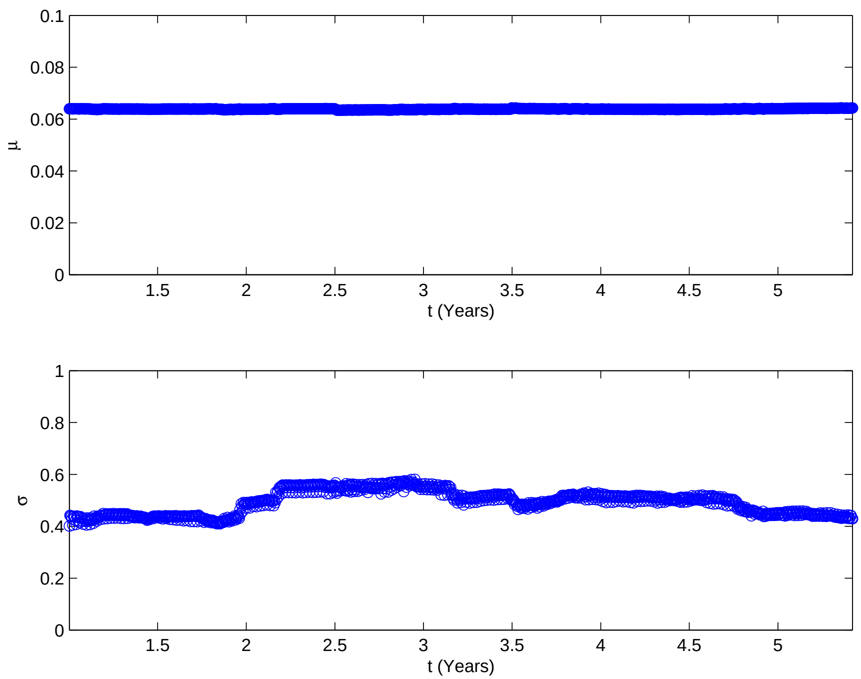
Let us begin showing that the relation established between the real data and the reconstructed parameters of the model is a stable relationship. The idea is the following.

We have more than 5 years of observations. We apply the calibration procedure on a large window of about one year of consecutive observation times.

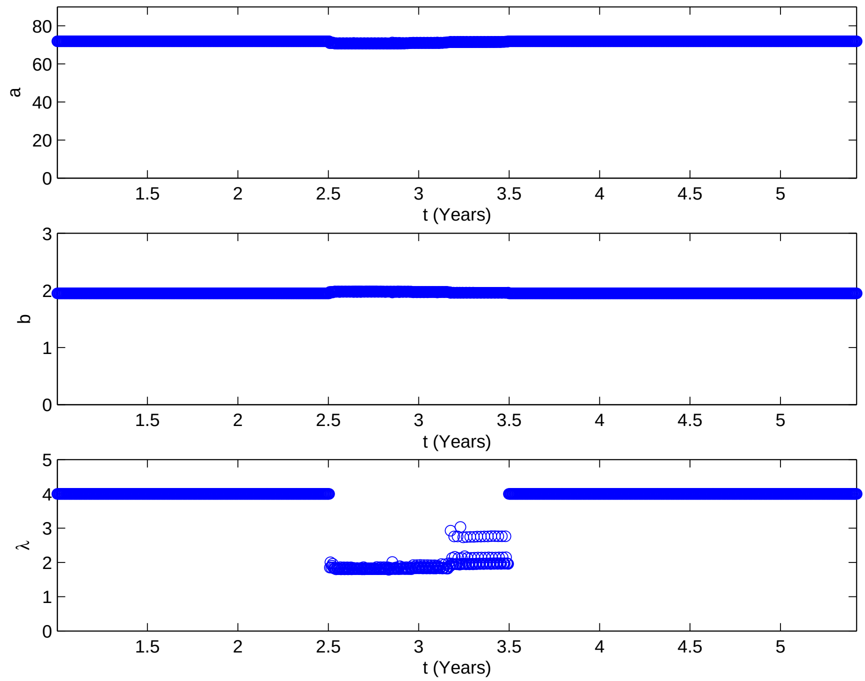
We move this window through the data time series discarding the data corresponding to the first observation time of the window and inserting the data corresponding to the next observation time after the window.

In this way we have about "four years of windows" and for each one of these windows we solve the corresponding calibration problem. We show that, changing the window, the reconstructed parameters remain stable.

The reconstructions of the parameters obtained moving the window along the data are shown below.



The Black-Scholes model parameters: μ, σ.

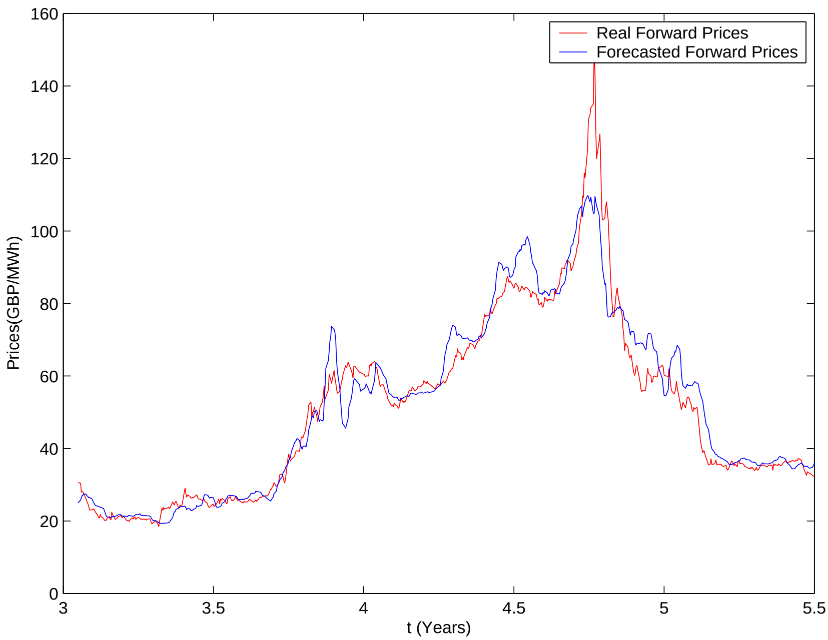


The spiky asset prices model parameters: a, b, λ.

2.2.9 Comparison Between the Real and the Forecasted Forward Electric Power Prices

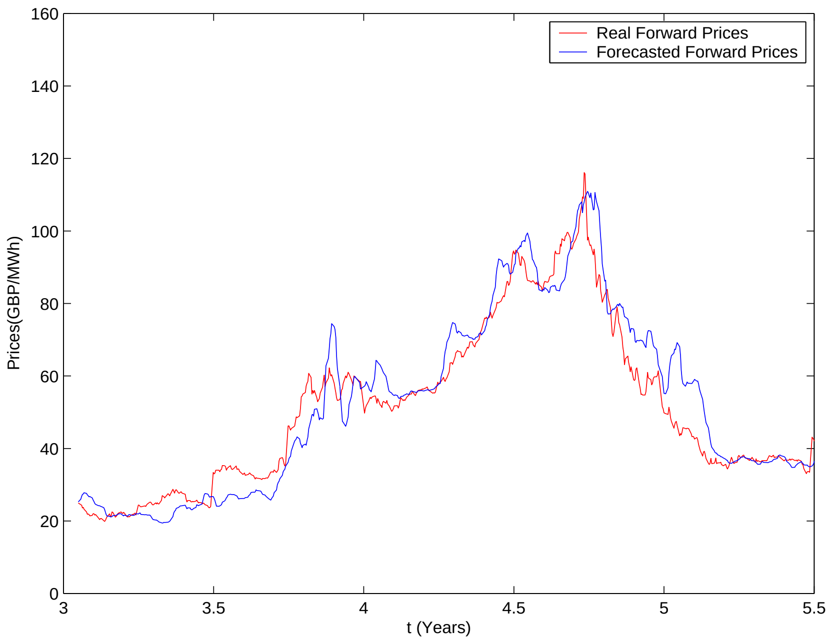
- A preliminary (ad hoc) step in the processing of real data is introduced and used to obtain a satisfactory formulation of the maximum likelihood problem as an optimization problem.
- The calibration procedure is applied to a large window of about three years of consecutive observation times.
- The following forward electric power prices are obtained.

2.2.10 Month-Ahead Prices



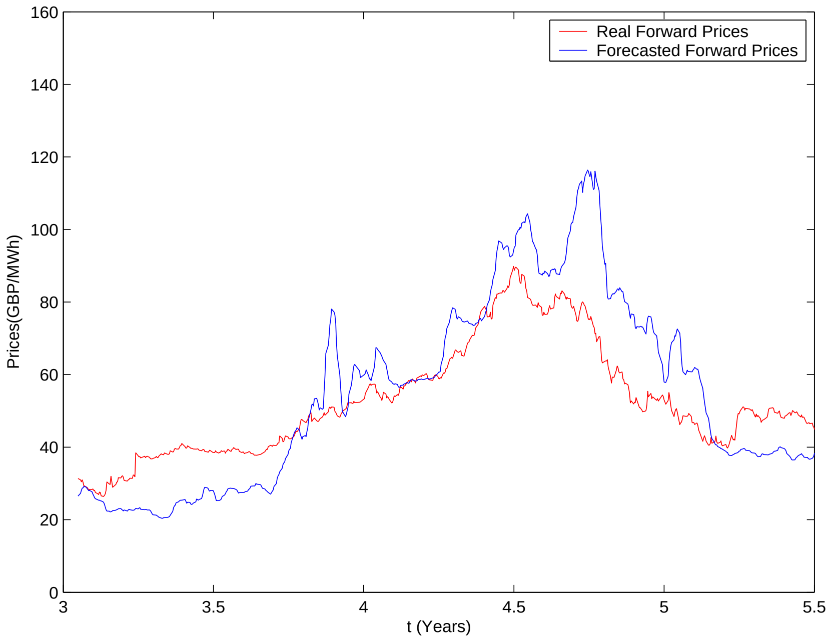
Relative error on the forecasted values 0.1179.

2.2.11 Quarter-Ahead Prices



Relative error on the forecasted values 0.1318.

2.2.12 1Year-Ahead Prices



Relative error on the forecasted values 0.2547.

2.2.13 References

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[3] Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2013). The analysis of real data using a multiscale stochastic volatility model, European Financial Management 19(1), 153-179.

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Section 2.3

The Analysis of Electric Power Price Data and of the S&P 500 Index Using a Multiscale Stochastic Volatility Model

[Description] *We use filtering and maximum likelihood methods to solve a calibration problem for a multiscale stochastic volatility model (including the risk premium parameters when necessary) and its two initial stochastic variances from the knowledge, at discrete times, of the asset price and, eventually, of the prices of call and/or put European options on the asset. This problem is translated in a maximum likelihood problem with the likelihood function defined through the solution of a filtering problem. We develop a tracking procedure that is able to track the asset price and the values of its two stochastic variances for time values where there are no data available. The solution of the calibration problem and the tracking procedure are used to do the analysis of data time series and to forecast asset and option prices. Specifically we study two time series of electric power price data taken from the U.S. electricity market and the 2005 data relative to the US S&P 500 index and to the prices of a call and a put European option on the S&P 500 index. The forecasts of the asset prices and of the option prices computed with the tracking procedure are compared with the prices actually observed and the comparison shows that they are of very high quality even when we consider "spiky" electric power price data.*

[Paper] *Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2013). The analysis of real data using a multiscale stochastic volatility model, European Financial Management 19(1), 153-179.*

[Website] *http://www.econ.univpm.it/recchioni/finance/w9*

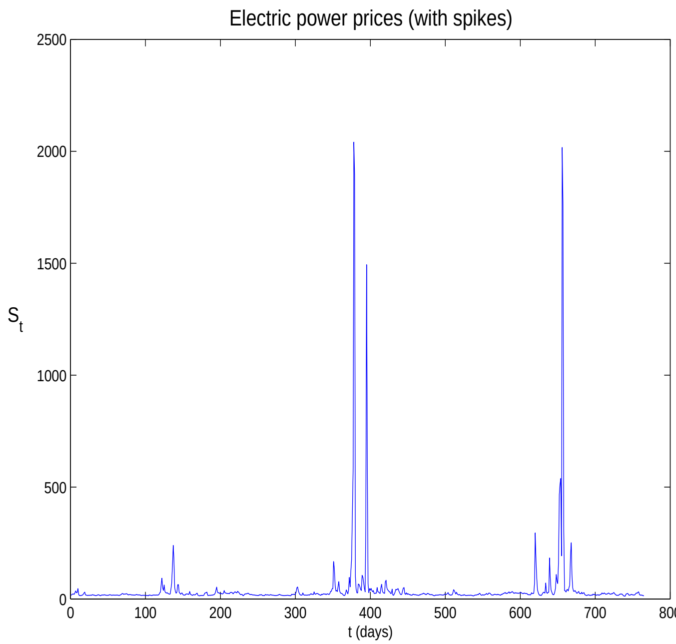
2.3.1 Outline of the Presentation

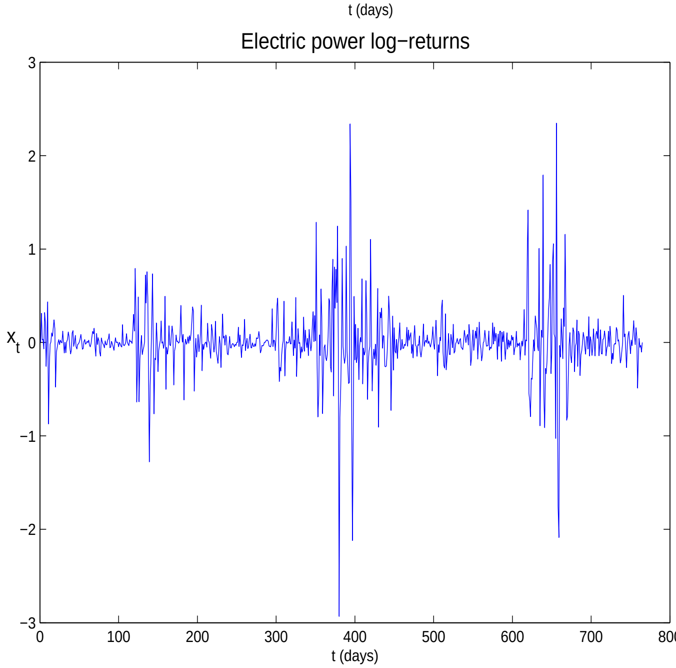
- Electric power price data
- The multiscale stochastic volatility model
- Formulae to forecast prices in the multiscale stochastic volatility model
- The calibration and filtering problems
- The maximum likelihood and the least squares approaches to the calibration problem
- Analysis of electric power prices and of the S&P500 index
- References

2.3.2 Notations

- S_t asset or commodity price at time t;
- μ drift rate of S_t;
- $x_t = \ln(S_t/S_0)$ log-return of the asset or of the commodity price at time t;
- $\tilde{C}_t$ observed price at time t of an European call option on the asset or commodity whose price is S_t with maturity time T and strike price E;
- $\tilde{P}_t$ observed price at time t of an European put option on the asset or commodity whose price is S_t with maturity time T and strike price E;
- $v_{i,t}$, $t > 0$, $i = 1, 2$, stochastic variances associated to the asset or commodity price S_t, $t > 0$.

2.3.3 Electric Power Prices and Associated Log-Return





The use of a (multiscale) stochastic volatility model to work with these prices is not only useful but it is necessary.

We choose the multiscale stochastic volatility model proposed by Fatone et al. 2009.

2.3.4 The Multiscale Stochastic Volatility Model

$$dx_t = (\mu + a_1 v_{1,t} + a_2 v_{2,t})dt + b_1\sqrt{v_{1,t}}dW_t^1 + b_2\sqrt{v_{2,t}}dW_t^2, t > 0,$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t})dt + \varepsilon_1\sqrt{v_{1,t}}dZ_t^1, \quad t > 0,$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t})dt + \varepsilon_2\sqrt{v_{2,t}}dZ_t^2, \quad t > 0,$$

$$x_0 = \tilde{x}_0, \quad v_{1,0} = \tilde{v}_{1,0}, \quad v_{2,0} = \tilde{v}_{2,0},$$

where the quantities a_i, b_i, χ_i, ε_i, θ_i, $i = 1, 2$, are real constants satisfying $\chi_i \geq 0$, $\varepsilon_i \geq 0$, $\theta_i \geq 0$, $\frac{2\chi_i\theta_i}{\varepsilon_i^2} > 1$, $i = 1, 2$. Moreover W_t^1, W_t^2, Z_t^1, Z_t^2, $t > 0$, are standard Wiener processes such that $W_0^1 = W_0^2 = Z_0^1 = Z_0^2 = 0$, dW_t^1, dW_t^2, dZ_t^1, dZ_t^2, $t > 0$, are their stochastic differentials.

The correlation structure of the model is given by:

$$\left(\begin{array}{c|cccc} & W^1 & Z^1 & W^2 & Z^2 \\ \hline W^1 & 1 & \rho_1 & 0 & 0 \\ Z^1 & \rho_1 & 1 & 0 & 0 \\ W^2 & 0 & 0 & 1 & \rho_2 \\ Z^2 & 0 & 0 & \rho_2 & 1 \end{array}\right)$$

2.3.5 The Multiscale Stochastic Volatility Model Generalizes the Heston Model

When we choose $a_1 = a_2 = -\frac{1}{2}$, $b_1 = b_2 = 1$, the model reduces to:

$$dx_t = (\mu - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t})dt + \sqrt{v_{1,t}}dW_t^1 + \sqrt{v_{2,t}}dW_t^2, t > 0,$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t})dt + \varepsilon_1\sqrt{v_{1,t}}dZ_t^1, \quad t > 0,$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t})dt + \varepsilon_2\sqrt{v_{2,t}}dZ_t^2, \quad t > 0,$$

$$x_0 = \tilde{x}_0, \quad v_{1,0} = \tilde{v}_{1,0}, \quad v_{2,0} = \tilde{v}_{2,0},$$

We call the model corresponding to this choice Double Heston model that generalizes the Heston model (see Heston 1993).

When $0 < \chi_1 < \chi_2$ the two stochastic variances $v_{1,t}$, $v_{2,t}$, $t > 0$, capture respectively the long term variance (slow time scale) and the short term variance (fast time scale). The model is multiscale when $0 < \chi_1 << \chi_2$.

2.3.6 Model Parameters

The parameter vector that must be estimated (from the observed data) is:

$$\underline{\Theta} = (\mu, \chi_1, \theta_1, \varepsilon_1, \tilde{v}_{1,0}, \lambda_1, \chi_2, \theta_2, \varepsilon_2, \lambda_2, \tilde{v}_{2,0}, \rho_1, \rho_2) \in \mathbb{R}^{13},$$

where λ_i, $i = 1, 2$ are the risk premium parameters.

Note that when we work on the calibration problem using as data only option prices we can incorporate the risk premium parameters λ_i, $i = 1, 2$, associated to the risk neutral measure into the parameters χ_i and θ_i, $i = 1, 2$.

That is a model with $\chi_i^* = \chi_i + \lambda_i$, $\theta_i^* = \theta_i\chi_i/(\chi_i + \lambda_i)$, instead of χ_i, θ_i,

$i = 1, 2$ should be considered to take into account the fact that the option prices are computed with respect to the risk neutral measure.

Note that when we work on the calibration problem using as data only asset prices we can omit the risk premium parameters λ_i, $i = 1, 2$.

2.3.7 Why do We Use a Multiscale Model?

1. Several empirical studies of real data have shown that the term structure of the implied volatility of the price of many underlyings seems to be driven by two different factors varying on two different time scales ($\chi_1 << \chi_2$).

2. This type of models is able to reproduce spikes through the use of a fast time scale volatility together with an intermediate time scale volatility.

Why do we use the previous multiscale model ?

1. The model contains as special cases some well known models such as the Black Scholes model and the Heston model.

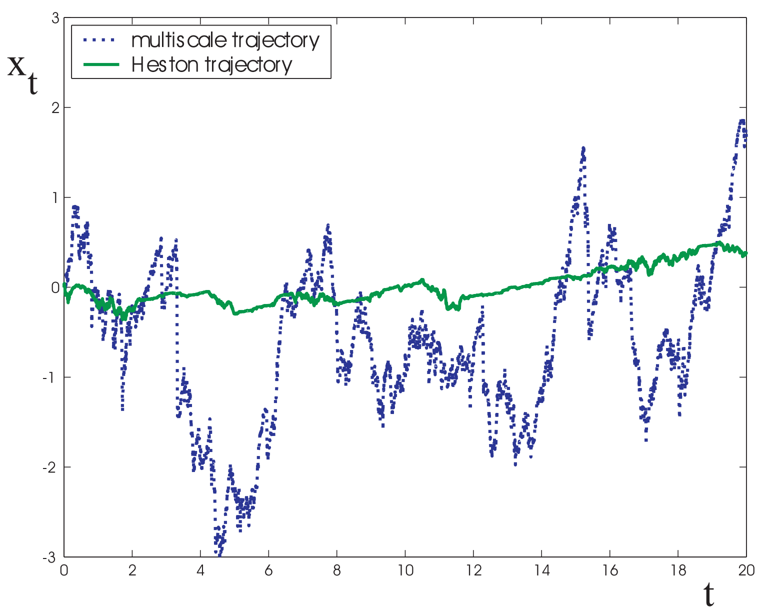
2. The model is explicitly solvable that is, under the assumptions made above on the correlation structure, the transition probability density function of the stochastic process solution of the model is represented as a one dimensional integral of an explicitly known integrand. This property makes possible to price put and call options in the model computing one dimensional integrals, that is using easy to handle formulae.

2.3.8 Spikes Generated by the Multiscale Stochastic Volatility Model

- Multiscale trajectory parameters: $\mu = 0.03$, $\theta_1 = 0.01$, $\theta_2 = 0.03$, $\chi_1 = 1$, $\chi_2 = 100$, $\rho_1 = -0.5$, $\rho_2 = -0.7$, $\varepsilon_1 = 0.25\sqrt{\chi_1}$, $\varepsilon_2 = 2\sqrt{\chi_2}$, $\tilde{v}_{1,0} = 0.05$, $\tilde{v}_{2,0} = 0.015$;

- Heston trajectory parameters: $\mu = 0.03$, $\theta_1 = 0.01$, $\chi_1 = 1$, $\rho_1 = -0.5$, $\varepsilon_1 = 0.25\sqrt{\chi_1}$, $\tilde{v}_{1,0} = 0.05$.

The choice $\chi_1 = 1$ and $\chi_2 = 100$ made in the multiscale trajectory parameters guarantees that the stochastic variances change on different time scales.



Example of synthetic data

2.3.9 An Explicitly Solvable Model

- The transition probability density function of the stochastic process solution of the model can be written as a one dimensional integral of an explicitly known integrand (Fatone at al. 2009).
- The price of European vanilla call and put options in the model can be written as a one dimensional integral of an explicitly known integrand (Fatone et al. 2009).
- The joint probability density function of the state variables of the model, that is of x_t and of the associated stochastic variances $v_{1,t}$, $v_{2,t}$ conditioned to the observations of the asset prices and of European options prices can be written as a double integral (Fatone et al. 2013).

2.3.10 One Dimensional Integral Formula for the Transition Probability Density Function of the Multiscale Model

We have derived the following formula:

$$p_f(x, v_1, v_2, t, x', v_1', v_2', t') = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x-x'-\mu\tau)} \cdot$$

$$\prod_{i=1}^{2} e^{-2\chi_i\theta_i((\nu_i+\zeta_i)\tau+\ln(s_{i,b}/(2\zeta_i)))/\varepsilon_i^2} \cdot$$

$$\left[e^{-2v_i'(\zeta_i^2-\nu_i^2)s_{i,g}/(\varepsilon_i^2 s_{i,b})} e^{-M_i(\tilde{v}_i+v_i)} M_i \left(\frac{v_i}{\tilde{v}_i}\right)^{(\chi_i\theta_i/\varepsilon_i^2)-1/2} \cdot \right.$$

$$\left. I_{2\chi_i\theta_i/\varepsilon_i^2-1}\left(2M_i(\tilde{v}_i v_i)^{1/2}\right)\right],$$

$$(x, v_1, v_2), (x', v_1', v_2') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+, t, t' \geq 0, t-t' > 0,$$

where the quantities $s_{i,b}$, $s_{i,g}$, $\tilde{v}_i$, M_i, $i = 1, 2$ are elementary functions.

2.3.11 The Elementary Functions Appearing in the Transition Probability Density Function

The functions $s_{i,b}$, $s_{i,g}$, $\tilde{v}_i$, M_i, $i = 1, 2$ are given by:

$$s_{i,g} = 1 - e^{-2\zeta_i\tau},\, s_{i,b} = \zeta_i - \nu_i + (\zeta_i + \nu_i)e^{-2\zeta_i\tau}, \tau > 0, i = 1, 2,$$

$$\tilde{v}_i = \frac{4v_i'\zeta_i^2 e^{-2\zeta_i\tau}}{(s_{i,b})^2},\;\; M_i = \frac{2s_{i,b}}{{\varepsilon_i}^2 s_{i,g}},\;\; \tau > 0,\; i = 1, 2,$$

where

$$\nu_i = -\frac{1}{2}\left(\chi_i + \imath\, k\, b_i\varepsilon_i\rho_i\right),\;\; k \in \mathbb{R},\; i = 1, 2,$$

$$\zeta_i = \frac{1}{2}\left(4\nu_i^2 + \varepsilon_i^2(b_i^2 k^2 + 2\imath\, k\, a_i)\right)^{1/2}, k \in \mathbb{R}, i = 1, 2.$$

2.3.12 European Vanilla Call Option Price in the Multiscale Stochastic Volatility Model

Using the risk neutral formula it can be seen that the price of a European vanilla call option at time $t = 0$ with time to maturity $\tau > 0$, strike price E and asset price S_0 at time $t = 0$ is:

$$C(\tau,E,S_0,\tilde{v}_{1,0},\tilde{v}_{2,0})=\frac{S_0}{2\pi}e^{-r\tau}e^{2\mu\tau}\int_{-\infty}^{+\infty}dk\,\frac{e^{-\imath k(\log(S_0/E)+\mu\tau)-\log(E/S_0)}}{-k^2-3\imath\,k+2}\cdot$$

$$\prod_{i=1}^{2}\left(e^{-2\chi_i^*\theta_i^*(\nu_i^c+\zeta_i^c+\log(s_{i,b}^c/(2\zeta_i^c)))\tau/\varepsilon_i^2}e^{-2\tilde{v}_{i,0}((\zeta_i^c)^2-(\nu_i^c)^2)s_{i,g}^c/(\varepsilon_i^2 s_{i,b}^c)}\right),$$

$$\tilde{v}_{1,0},\tilde{v}_{2,0}>0,$$

where r is the risk free interest rate, $\tilde{v}_{1,0}$, $\tilde{v}_{2,0}$ are the stochastic variances at time $t = 0$ that cannot be observed in real markets and that must be estimated from price data. Finally we have:

$$\nu_i^c = -\frac{1}{2}\left(\chi_i^* + \imath\,k\,b_i\varepsilon_i\rho_i - 2b_i\rho_i\varepsilon_i\right), k \in \mathbb{R}, i = 1, 2,$$

$$\zeta_i^c=\frac{1}{2}\left(4(\nu_i^c)^2+\varepsilon_i^2(b_i^2k^2+2\imath\,k\,a_i+4\imath\,kb_i^2-4(a_i+b_i^2))\right)^{1/2},$$

$$k \in \mathbb{R}, i = 1, 2,$$

$$s_{i,g}^c=1-e^{-2\zeta_i^c\tau}, s_{i,b}^c=\zeta_i^c-\nu_i^c+(\zeta_i^c+\nu_i^c)e^{-2\zeta_i^c\tau}, \tau>0, i=1,2.$$

A similar formula holds for the put option price.

2.3.13 Conditioned Joint Probability Density Function

Let be $t_0 = 0$, $t_i < t_{i+1}$, $i = 0, 1, \ldots, n-1$.

We suppose that the option price observations $\tilde{C}_i$, $\tilde{P}_i$, made at time $t = t_i$,$i = 0, 1, \ldots, n$, are affected by a Gaussian error with mean zero and known variance ϕ_i, $i = 0, 1, \ldots, n$, and that the asset log-returns $\tilde{x}_i$, $i = 0, 1, \ldots, n$, are observed without error.

The joint probability density function of x_t and of the associated stochastic variances $v_{1,t}$, $v_{2,t}$ conditioned to the observations $\mathcal{F}_t = \{(\tilde{x}_i, \tilde{C}_i, \tilde{P}_i) : t_i \leq t, i > 0\}$, $t > 0$, can be represented as follows:

$$p_i(x, v_1, v_2, t|\underline{\Theta}) = \int_0^{+\infty} dv_1' \int_0^{+\infty} dv_2' p_f(x, v_1, v_2, t, \tilde{x}_i, v_1', v_2', t_i) \, f_i(v_1', v_2'; \underline{\Theta}),$$

$$(x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \; t_i < t < t_{i+1}, i = 0, 1, \ldots, n,$$

$$p_i(x, v_1, v_2, t_i|\underline{\Theta}) = \delta(x - \tilde{x}_i) f_i(v_1, v_2; \underline{\Theta}), \; (x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \; i = 0, 1, \ldots, n,$$

where $f_i(v_1', v_2'; \underline{\Theta})$, $i = 0, 1, \ldots, n$ are given below.

2.3.14 The Functions f_i, $i = 0, 1, \ldots, n$

$$f_0(v_1, v_2; \underline{\Theta}) = \delta(v_1 - \tilde{v}_{1,0})\delta(v_2 - \tilde{v}_{2,0}), \; (v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

and for $i = 1, 2, \ldots, n$:

$$f_i(v_1, v_2; \underline{\Theta}) = \frac{p_{i-1}(\tilde{x}_i, v_1, v_2, t_i^-|\underline{\Theta})\pi_1(\tilde{x}_i, v_1, v_2, t_i|\underline{\Theta})}{\int_0^{+\infty}\int_0^{+\infty} p_{i-1}(\tilde{x}_i, v_1', v_2', t_i^-|\underline{\Theta})\pi_1(\tilde{x}_i, v_1', v_2', t_i|\underline{\Theta}) dv_1' dv_2'},$$

$$(x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,$$

$$\pi_1(\tilde{x}_i, v_1, v_2, t_i|\underline{\Theta}) = \frac{1}{\sqrt{2\pi\phi_i}} \frac{1}{\sqrt{2\pi\phi_i}} \cdot e^{\left(-\frac{1}{2\phi_i}\left[(\tilde{C}_i - C(\tilde{x}_i, v_1, v_2, t_i; K_i, T_i, \underline{\Theta}))^2 + (\tilde{P}_i - P(\tilde{x}_i, v_1, v_2, t_i; K_i, T_i, \underline{\Theta}))^2\right]\right)},$$

$$(\tilde{x}_i, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,$$

where $p_{i-1}(\tilde{x}_i, v_1, v_2, t_i^-|\underline{\Theta}) = \lim_{t\to t_i^-} p_{i-1}(\tilde{x}_i, v_1, v_2, t|\underline{\Theta})$, $\lim_{t\to t_i^-}$ means left limit for t that goes to t_i, $i = 1, 2, \dots, n$.

2.3.15 Formulae to Forecast the Log-Return and the Associated Variances

Given the parameter vector $\underline{\Theta}$ of the stochastic model, we can forecast the values of the state variables of the model x_t, $v_{1,t}$, $v_{2,t}$, $t > 0$, respectively as the expected values $\hat{x}_{t|\underline{\Theta}}$, $\hat{v}_{1,t|\underline{\Theta}}$, $\hat{v}_{2,t|\underline{\Theta}}$, $t > 0$, conditioned to the observations contained in $\mathcal{F}_t$, $t > 0$, of the random variables x_t, $v_{1,t}$, $v_{2,t}$, $t > 0$. That is for $t_i \le t < t_{i+1}$, $i = 0, 1, \dots, n$ we have:

$$\begin{aligned}
\hat{x}_{t|\underline{\Theta}} &= \mathbb{E}(x_t|\mathcal{F}_{t_i}, \underline{\Theta}) \\
&= \tilde{x}_i + (\mu - \frac{\theta_1}{2} - \frac{\theta_2}{2})(t - t_i) + \sum_{j=0}^{1}\Bigg\{\theta_{2-j}\left(\frac{1 - e^{-\chi_{2-j}(t-t_i)}}{2\chi_{2-j}}\right) \\
&-\frac{(1 - e^{-\chi_{2-j}(t-t_i)})}{2\chi_{2-j}} \int_0^{+\infty} dv_{2-j} v_{2-j} \int_0^{+\infty} dv_{j+1} f_i(v_1, v_2; \underline{\Theta})\Bigg\}\ \hat{v}_{j,t|\underline{\Theta}} \\
&= \mathbb{E}(v_{j,t}|\mathcal{F}_{t_i}, \underline{\Theta}) = \theta_j(1 - e^{-\chi_j(t-t_i)})) \\
&+ e^{-\chi_j(t-t_i)} \int_0^{+\infty}\int_0^{+\infty} dv_1 dv_2\, v_j f_i(v_1, v_2; \underline{\Theta}),\ j = 1, 2,
\end{aligned}$$

2.3.16 Calibration and Filtering Problems

We want to estimate the parameter vector $\underline{\Theta}$ of the multiscale model starting from price data. To this aim we solve the following problems:

1. Calibration (Estimation) Problem: find an estimate of the vector $\underline{\Theta}$ starting from the observations, that is, for example, from the

knowledge at time $t = t_i$ ($t_i < t_{i+1}, t_{n+1} = +\infty$) of the stock log-return $\tilde{x}_i$ and/or of a call option price $\tilde{C}_i$,, and/or of a put option price $\tilde{P}_i$ for $i = 0, 1, \ldots, n$. This means find the value of the vector $\underline{\Theta}$ that makes most likely the observations $\mathcal{F}_t = \{(\tilde{x}_i, \tilde{C}_i, \tilde{P}_i) : t_i \leq t\}, t > t_0$.

2. Filtering Problem (Forecasting Problem): given the value of the model parameter vector $\underline{\Theta}$ forecast the stock log-return for $t \neq t_i$, $i = 1, 2, \ldots, n$ and the stochastic variances $v_{1,t}$, $v_{2,t}$ for $t \neq t_i$, and in particular for $t > t_n$. The filtering problem has been solved in the previous slide and the probability density function conditioned to the observations employed in its solution is used in the solution of the calibration problem.

2.3.17 Calibration Problem - Maximum Likelihood Approach (ML)

Let $\mathbb{R}^{13}$ be the 13 dimensional real Euclidean vector space and let $\mathcal{M}$ be the set of the admissible vectors $\underline{\Theta}$, that is:

$$\mathcal{M} = \{\underline{\Theta} = (\epsilon_1, \theta_1, \rho_{0,1}, \chi_1, \tilde{v}_{0,1}, \mu, \lambda_1, \epsilon_2, \theta_2, \rho_{0,2}, \chi_2, \tilde{v}_{0,2}, \lambda_2)$$
$$\in \mathbb{R}^{13} \,|\, \varepsilon_i\, \chi_i, \theta_i \geq 0, i = 1, 2, \frac{2\chi_i \theta_i}{\varepsilon_i^2} \geq 1,$$
$$-1 \leq \rho_{0,i} \leq 1, \tilde{v}_{0,i} \geq 0,\ \chi_i + \lambda_i > 0,\ i = 1, 2\}.$$

The solution of the following maximum likelihood problem is a point $\underline{\Theta}^*$ whose coordinates are the parameter values that "make most" likely the occurrence of the observations $(\tilde{x}_i, \tilde{C}_i, \tilde{P}_i)$ at time $t = t_i$, $i = 0, 1, \ldots, n$:

$$\max_{\underline{\Theta} \in \mathcal{M}} F(\underline{\Theta}),$$

where $F(\underline{\Theta})$ is the (log-)likelihood function defined as follows:

$$F(\underline{\Theta}) = \sum_{i=0}^{n-1} \log\Big[\int_0^{+\infty}\int_0^{+\infty} p_i(\tilde{x}_{i+1}, v_1, v_2, t_{i+1}^-|\underline{\Theta})$$

$$\pi_1(\tilde{x}_{i+1}, v_1, v_2, t_{i+1}|\underline{\Theta})dv_1dv_2\Big] +$$

$$\log[\pi_1(\tilde{x}_0, \tilde{v}_{1,0}, \tilde{v}_{2,0}, t_0|\underline{\Theta})], \quad \underline{\Theta} \in \mathcal{M}^*.$$

2.3.18 Calibration Problem - Least Squares Approach (LS)

Let $\mathbb{R}^{11}$ be the 11 dimensional real Euclidean vector space and let $\mathcal{M}$ be the set of the admissible vectors $\underline{\Theta}$, that is:

$$\mathcal{M}^* = \{\underline{\Theta} = (\epsilon_1, \theta_1^*, \rho_1, \chi_1^*, \tilde{v}_{0,1}, \mu, \epsilon_2, \theta_2^*, \rho_2, \chi_2^*, \tilde{v}_{0,2}) \in \mathbb{R}^{11} \,|$$

$$\varepsilon_i, \chi_i^*, \theta_i^* \geq 0,\, i=1,2,\, \frac{2\chi_i^*\theta_i^*}{\varepsilon_i^2} \geq 1, -1 \leq \rho_i \leq 1, \tilde{v}_{0,i} \geq 0, i=1,2\},$$

at time t, $t \geq 0$, we solve the following optimization problem:

$$\min_{\underline{\Theta}\in\mathcal{M}^*} L_t(\underline{\Theta}),$$

where the objective function $L_t(\underline{\Theta})$, $t \geq 0$, is defined as follows:

$$L_t(\underline{\Theta}) = \sum_{i=1}^{m_c}\Big[C^{t,\underline{\Theta}}(\tilde{S}_t, T_i, K_i) - \tilde{C}^t(\tilde{S}_t, T_i, K_i)\Big]^2 +$$

$$\sum_{i=1}^{m_p}\Big[P^{t,\underline{\Theta}}(\tilde{S}_t, T_i, K_i) - \tilde{P}^t(\tilde{S}_t, T_i, K_i)\Big]^2,$$

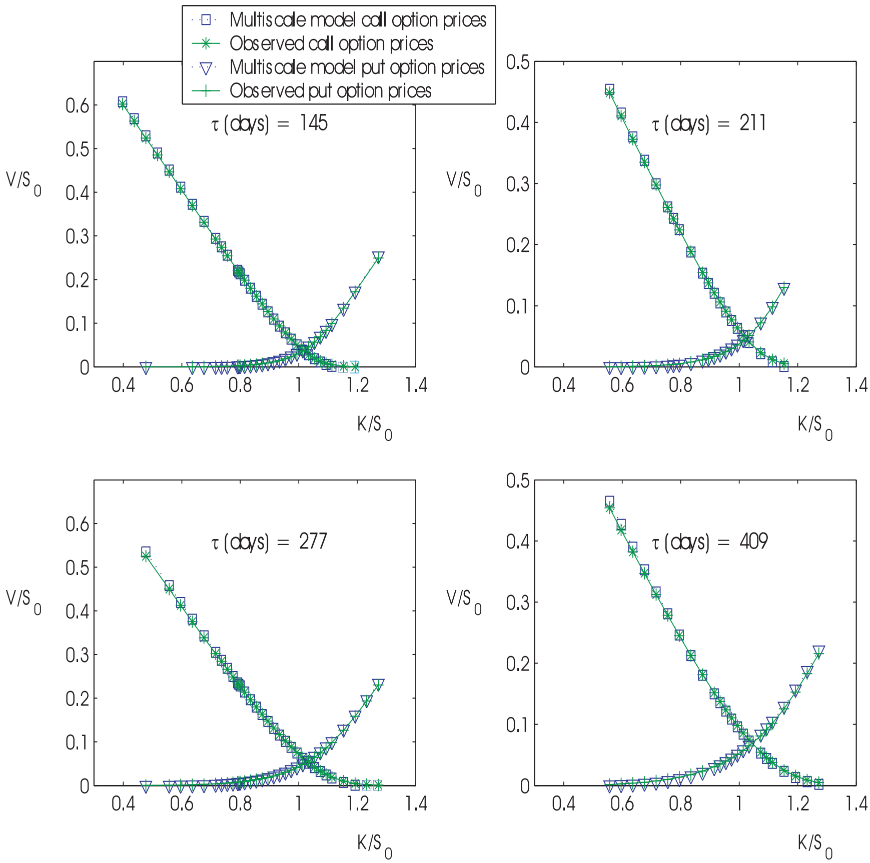
$\tilde{C}^t$, $\tilde{P}^t$ are the observed prices (data) of European vanilla call and put options respectively and $C^{t,\underline{\Theta}}$, $P^{t,\underline{\Theta}}$ are the corresponding theoretical prices. Note only option prices are used as data in this formulation of the calibration problem.

2.3.19 Numerical Results on Real Data: S&P 500 Index

1. LS approach. For each month we proceed solving the calibration problem using all (in, at, out of the money) the call and put option (daily closing) prices available to us relative to the third day of the month. For example November 3, 2005 ($m_c = 303$ call options prices and $m_p = 284$ put options prices). The implied values of the vector $\underline{\Theta}$ obtained solving the calibration problem using the data of November 3, 2005 (i.e. the third day of the month) are used to forecast the option prices of November 7 ($m_c = 303$, $m_p = 290$), November 14 ($m_c = 305$, $m_p = 295$), and November 28 ($m_c = 292$, $m_p = 265$), 2005. This procedure is used for each month considered. The total number of data used in each calibration problem is approximately $5 - 600$.

2. ML approach. For each month where we want to forecast the S&P500 log-return we use the data contained in a window made of the last fifteen consecutive observation days of the previous month. For example to forecast the value of the log return in November 7, 14, 28, 2005 we use as data the last fifteen daily observations (daily closing values) of October 2005 of the log-return of the S& P 500 index and of the call and put option (bid) prices on the $S\&P$ 500 index having maturity time December 16, 2005 and strike price $E = 1200$. The total number of data used is $15 \times 3 = 45$.

2.3.20 Forecasted Values of Call and Put Options



November 28, 2005: European vanilla call and put option prices (V) on the $S\&P500$ index forecasted using the multiscale model and prices observed in the market (model calibration done with the data of November 3, 2005) versus moneyness K/S_0. The value of the S&P500 of November 28, 2005 is assumed to be known when the option prices are forecasted.

2.3.21 Maximum Likelihood Calibration - Errors on Forecasts

$number\, of\, days in\, the$ $future\, of\, the\, forecast$	e_{index}	$e_{call\, option}$	$e_{put\, option}$
1	$7.522 \cdot 10^{-5}$	0.0659	0.0407
2	$1.303 \cdot 10^{-4}$	0.0559	0.0737
3	$1.7509 \cdot 10^{-4}$	0.0893	0.0952
4	$2.6962 \cdot 10^{-4}$	0.0528	0.0793
5	$2.9114 \cdot 10^{-4}$	0.0496	0.0822
15	$3.7106 \cdot 10^{-4}$	0.0268	0.1810
30	$3.5094 \cdot 10^{-4}$	0.0717	0.1099

Average of the relative errors of the forecasted values of the SP&500 and of the corresponding call and put option prices having strike price $E = 1200$ and maturity time $T = \text{December}\, 16, 2005$ in January and February 2005 when compared to the prices actually observed. Note that the forecasts of the option prices are made using the forecasted values of the S&P500 index.

2.3.22 Comparison Between Least Squares (LS) Approach and Maximum Likelihood (ML) Approach

The quality of the forecasted values of the option prices is established comparing the prices actually observed with the forecasted prices when the maximum likelihood (ML) or the least squares (LS) method are used in the calibration of the multiscale model.

Date t	$\epsilon^t_{mean,ML}$	$\epsilon^t_{mean,LS}$
January 28, 2005	$4.69 \cdot 10^{-3}$	$2.84 \cdot 10^{-3}$
June 7, 2005	$5.99 \cdot 10^{-3}$	$1.75 \cdot 10^{-3}$
June 28, 2005	$6.87 \cdot 10^{-3}$	$2.56 \cdot 10^{-3}$
November 7, 2005	$3.68 \cdot 10^{-3}$	$3.04 \cdot 10^{-3}$
November 14, 2005	$3.07 \cdot 10^{-3}$	$2.21 \cdot 10^{-3}$
November 28, 2005	$3.21 \cdot 10^{-3}$	$2.41 \cdot 10^{-3}$

- The ML approach uses as data only fifteen values (observed in fifteen consecutive days) of a unique option price (call or put) (exercise price $E = 1200$, maturity date $Decem\ ber 16, 2005$) and of the S&P 500 log-return.

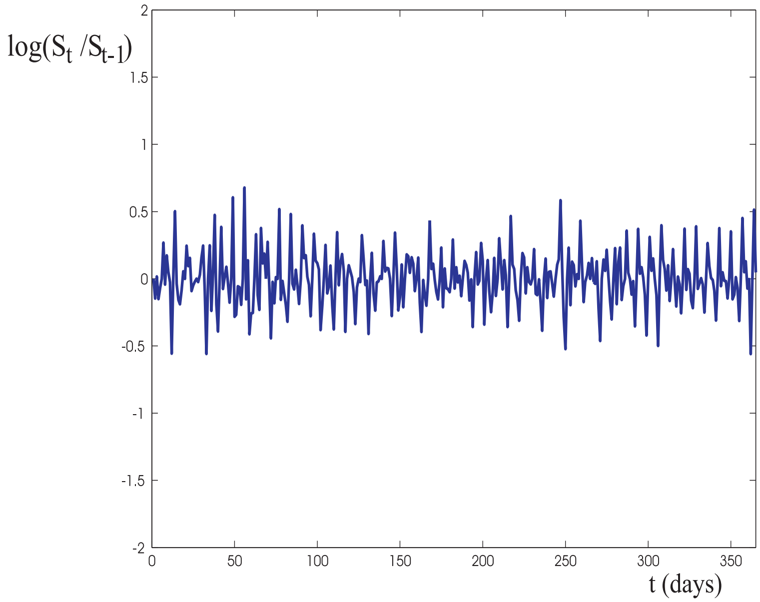
- The LS approach uses as data about 5-600 option prices (both calls and puts) observed in a given day.

Remember that the calibration in the LS approach is made assuming known the data up to the third day of the month where we do the forecasts.

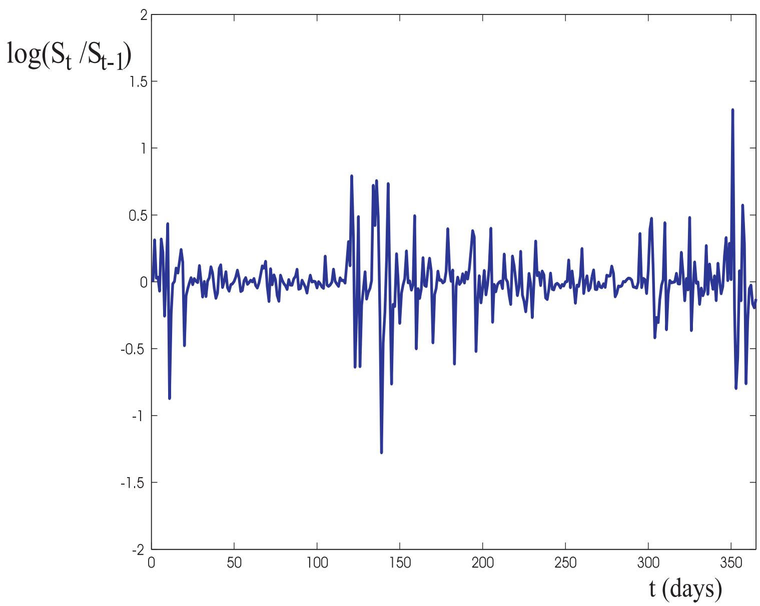
2.3.23 Analysis of Electric Power Prices

The numerical experiment considers two time series of electric power price data. The first time series consists of 365 daily observations $\tilde{S}_i$, $i = 0, 1, \ldots, 364$, that is it is a year, made of 365 days, of daily observations. The second time series consists of 765 daily observations $\hat{S}_i$, $i = 0, 1, \ldots 764$. In this experiment no option price data are used. The following Figures show the daily log-return increment

$\tilde{x}_i - \tilde{x}_{i-1} = \log(\tilde{S}_i/\tilde{S}_{i-1})$, $i = 1, 2, \ldots, 364$, of the electric power price data $\tilde{S}_i$, $i = 0, 1, \ldots, 364$, and the log-return increment of the electric power price data $\hat{S}_i$, $i = 0, 1, \ldots, 364$.



(a)

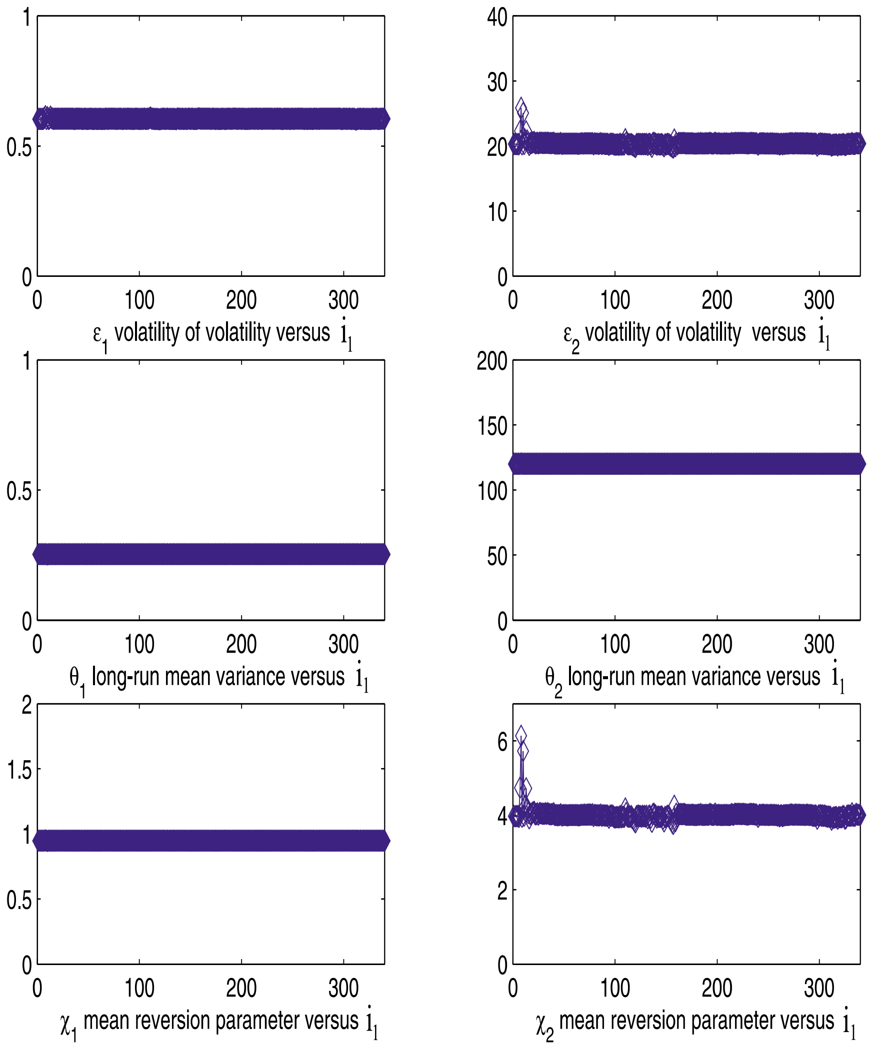


(b)

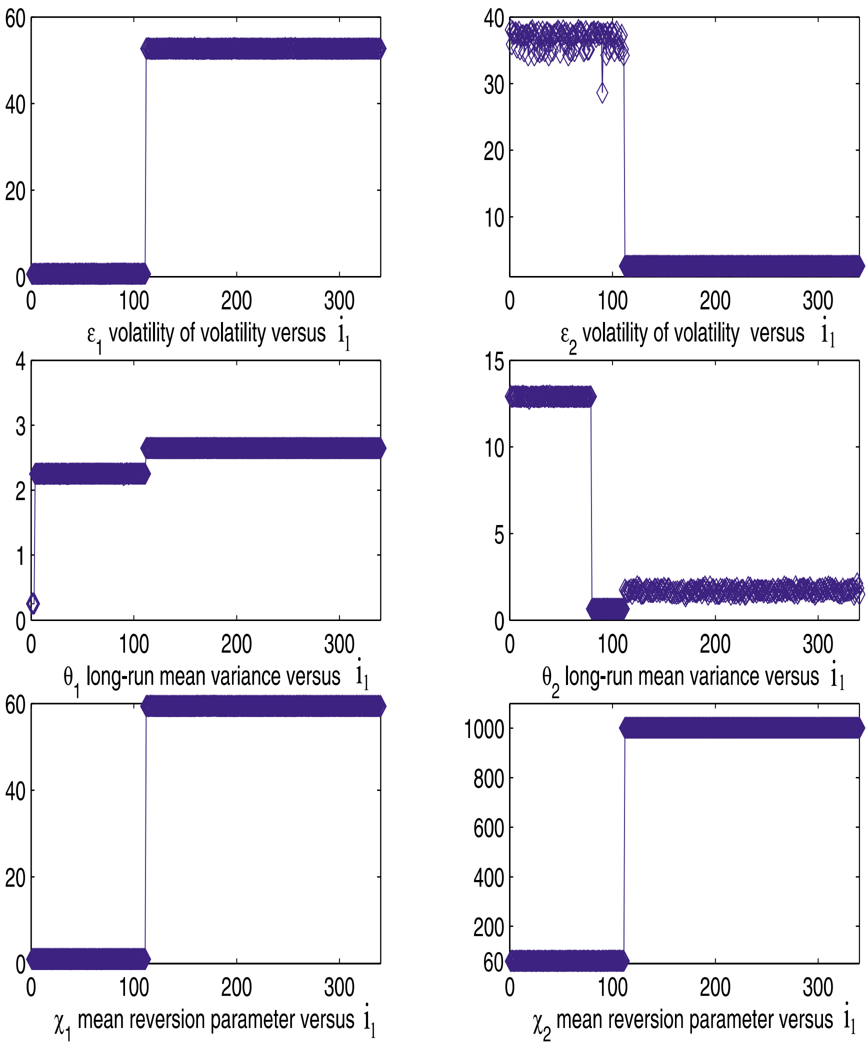
Electric log-return increment series ($\tilde{S}_i$) with no spikes (a) electric log-return increment series ($\hat{S}_i$) with spikes (b).

2.3.24 Analysis of Electric Power Prices: Numerical Results

We begin the analysis of these time series calibrating the multiscale model using a data window made of 26 consecutive daily observations and we move this window along the time series substituting the first observation of the window with the next observation after the window. Figures (c) and (d) show the results obtained solving the 340(=365-26+1) calibration problems associated to the two time series (Figures (a) and (b)) of data as a function of the index i_1 for $i_1 = 1, 2, \ldots, 340$. The index i_1 is the index associated to the first observation day of the data window used in the calibration.


ε1 volatility of volatility versus i1
ε2 volatility of volatility versus i1
θ1 long-run mean variance versus i1
θ2 long-run mean variance versus i1
χ1 mean reversion parameter versus i1
χ2 mean reversion parameter versus i1

(c)

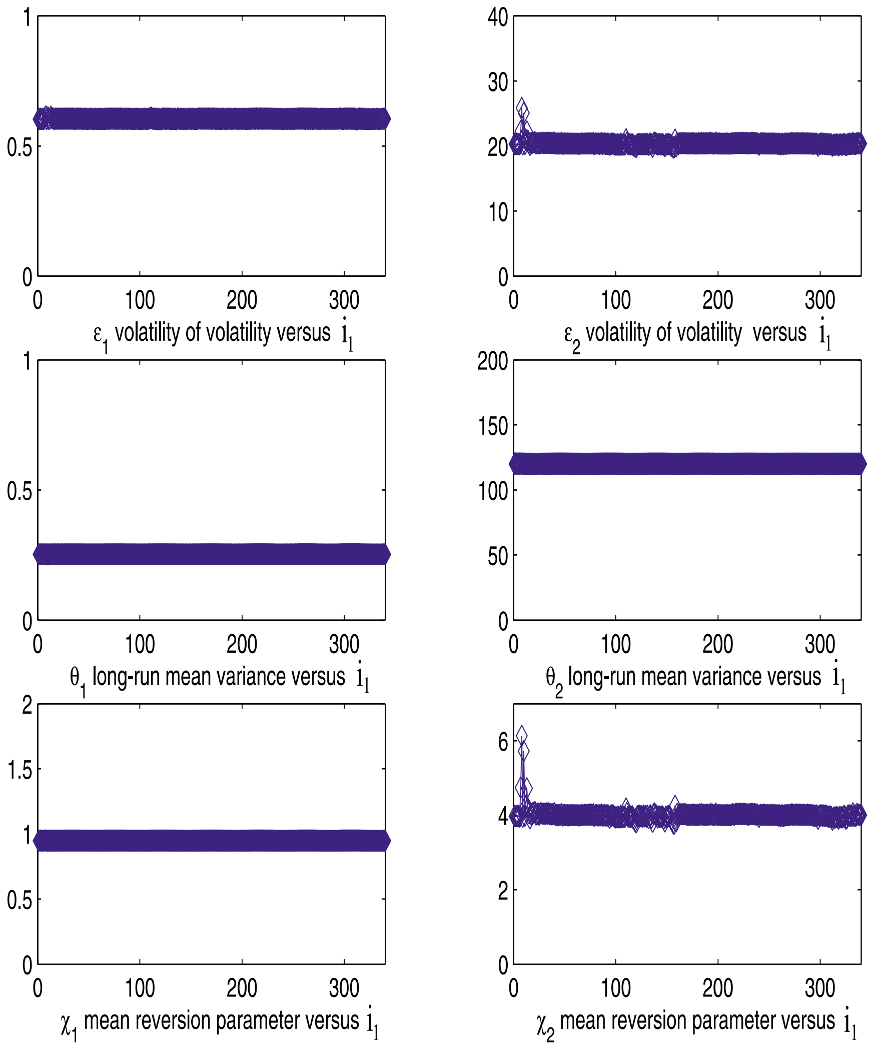


(d)

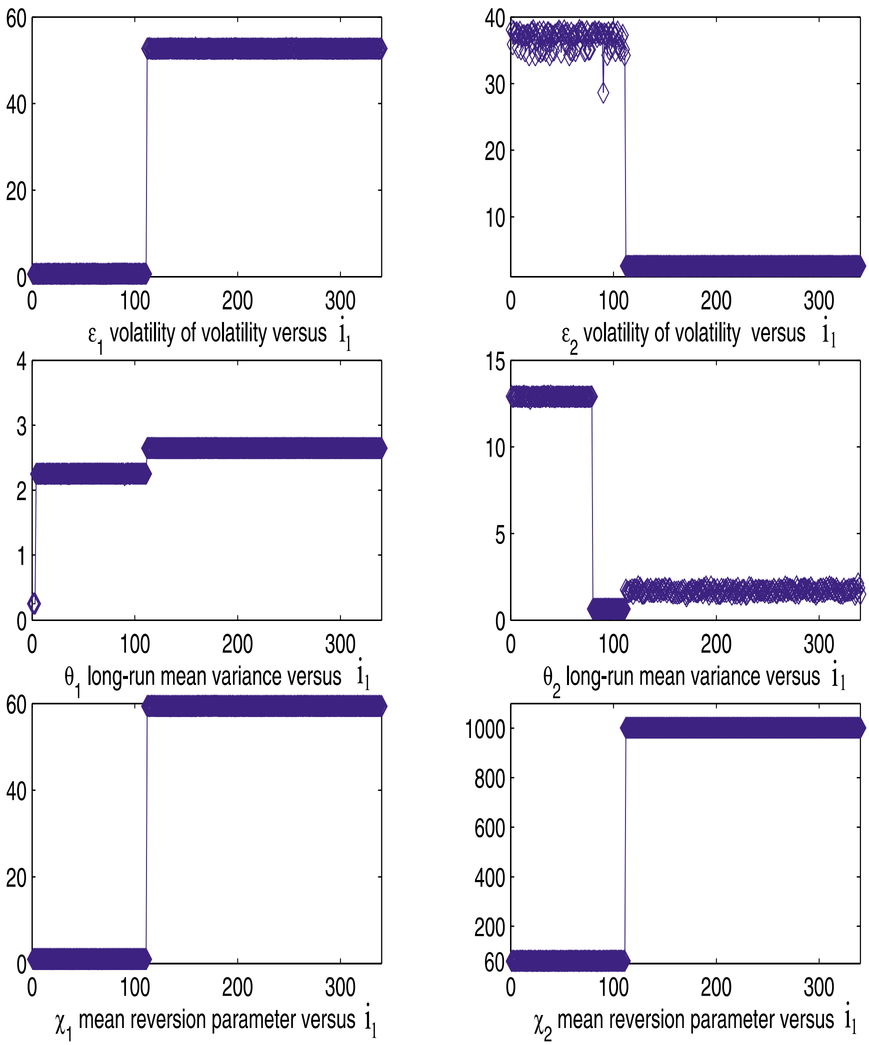
Estimated parameter values obtained solving 341 calibration problems using as data the electric power prices (time series with no spikes, Figures (a),(c)) and (time series with spikes, Figures (b),(d)) versus the calibration problem number i_1.

2.3.25 Analysis of Electric Power Prices: Description of the Results

We focus our attention on Figures (c) and (d) to point out the different behaviour of the estimated parameter values in the two cases. In absence of spikes (Figure (c)) the parameters ϵ_i, θ_i, χ_i, $i = 1, 2$, are stable, that is they are approximately constants as a function of i_1, and χ_1 and χ_2 are of the same order of magnitude (i.e.: $\chi_1 \approx 1$, $\chi_2 \approx 4$). In presence of spikes (Figure (d)) the parameters ϵ_i, θ_i, χ_i, $i = 1, 2$, have a jump in correspondence of the first spike. Moreover looking Figure (b) and Figure (d) we can see that the calibration procedure produces two values of the ratio χ_1/χ_2, in particular after the spike (i.e. $i_1 > 100$) there is a jump in the ratio χ_1/χ_2 and we have $\chi_1 << \chi_2$.


ε1 volatility of volatility versus i1
ε2 volatility of volatility versus i1
θ1 long-run mean variance versus i1
θ2 long-run mean variance versus i1
χ1 mean reversion parameter versus i1
χ2 mean reversion parameter versus i1

(c)

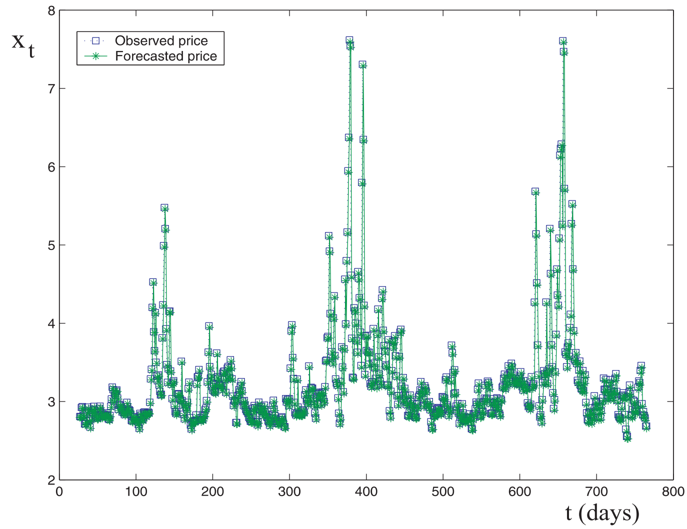


(d)

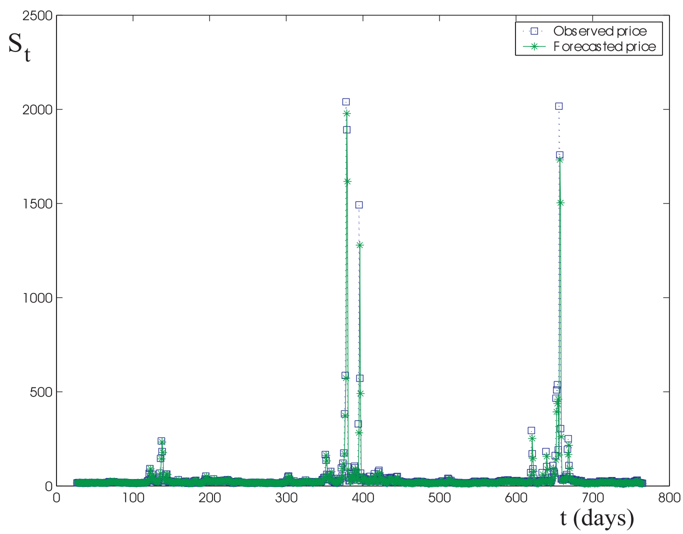
Estimated parameter values (time series with no spikes, Figure (c)) and (time series with spikes, Figure (d)) versus the calibration problem number i_1.

2.3.26 Analysis of Electric Power Prices: Forecasted Prices

Using the formulae presented above to forecast the log-return x_t and the corresponding price S_t when we use as data the electric power prices $\hat{S}_i$, $i = 0, 1, \ldots, 764$ (time series with spikes) we obtain the results shown in the Figures below.



(a)



(b)

Forecasted values (one day in the future) (stars) and observed values (squares) of the log-returns (a) and corresponding forecasted values (one day in the future) (stars) and observed values (squares) of electric power prices (b).

Forecasted values and observed values one day in the future (Movie)

2.3.27 Future Work

- Use the multiscale stochastic volatility model to evaluate insurance products such as life insurance products. That is we want to apply the approach proposed here, typical of quantitative finance, to the problem of pricing insurance products coupling to the stochastic equations that define the multiscale model (used to describe, for example, the S&P 500 that is the financial part of the life insurance product) a new stochastic differential equation that models some typical insurance variable, for example, a demographic variable (such as mortality).
- Develop efficient grid enabled algorithms to solve the optimization problems coming from the application of the maximum likelihood approach to the calibration problem.

Note that several numerical experiments and digital movies relative to the problems considered here can be found in the website:
http://www.econ.univpm.it/recchioni/finance/w8.

A more general reference to the work in mathematical finance of the authors and of their coauthors is the website:
http://www.econ.univpm.it/recchioni/finance.

2.3.28 References

[1] Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2009). An explicitly solvable multi-scale stochastic volatility model: option pricing and calibration, Journal of Futures Markets 29(9), 862-893.

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[3] Heston S. (1993). A closed form solution for options with stochastic volatility with applications to bond and currency options, Review of Financial Studies 6, 327-343.