Chapter 3

On Paranorm Zweier I-Convergent Sequence Spaces

“There is no place in the world for ugly mathematics. It may be very hard to define mathematical beauty but that is just as true of beauty of any kind , we may not know quite, what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it.”-Hardy

3.1 Introduction

The following subspaces of ω were first introduced and discussed by Maddox [56] :

$l(p) := \{x \in \omega : \sum\limits_{k} |x_k|^{p_k} < \infty\},$

$l_\infty(p) := \{x \in \omega : \sup\limits_{k} |x_k|^{p_k} < \infty\},$

$c(p) := \{x \in \omega : \lim\limits_{k} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\ \},$

$c_0(p) := \{x \in \omega : \lim\limits_{k} |x_k|^{p_k} = 0,\ \},$

where $p = (p_k)$ is a sequence of strictly positive real numbers.

After then Lascarides[53-54] defined the following sequence spaces :

$l_\infty\{p\} := \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup\limits_{k} |x_k r|^{p_k} t_k < \infty\},$

$c_0\{p\} := \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim\limits_{k} |x_k r|^{p_k} t_k = 0,\ \},$

$l\{p\} := \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum\limits_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\},$

Where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

Recently Khan and Ebadullah [38] introduced the following classes of sequence spaces:

$$\mathcal{Z}^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = L \text{ for some L}\} \in I\};$$

$$\mathcal{Z}_0^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = 0\} \in I\};$$

$$\mathcal{Z}_\infty^I = \{(x_k) \in \omega : \sup_k |Z^p x| < \infty\}.$$

We also denote by

$$m^I_{\mathcal{Z}} = \mathcal{Z}_\infty \cap \mathcal{Z}^I;$$

and

$$m^I_{\mathcal{Z}_0} = \mathcal{Z}_\infty \cap \mathcal{Z}^I_0.$$

In this chapter we introduce the following classes of sequence spaces:

$$\mathcal{Z}^I(q) = \{(x_k) \in \omega : \{k \in \mathbb{N} : |Z^p x - L|^{q_k} \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C} \ \};$$

$$\mathcal{Z}^I_0(q) = \{(x_k) \in \omega : \{k \in \mathbb{N} : |Z^p x|^{q_k} \geq \epsilon\} \in I\};$$

$$\mathcal{Z}^I_\infty(q) = \{(x_k) \in \omega : \sup_k |Z^p x|^{q_k} < \infty\}.$$

We also denote by

$$m^I_{\mathcal{Z}}(q) = \mathcal{Z}^I_\infty(q) \cap \mathcal{Z}^I(q);$$

and

$$m^I_{\mathcal{Z}_0}(q) = \mathcal{Z}^I_\infty(q) \cap \mathcal{Z}^I_0(q);$$

where $q = (q_k)$, is a sequence of positive real numbers.

Throughout the chapter, for the sake of convenience we will denote by $Z^p x = x^/, Z^p y = y^/, Z^p z = z^/$ for all $x, y, z \in \omega$.

3.2 Main Results

Theorem 3.2.1. The classes of sequences $\mathcal{Z}^I(q), \mathcal{Z}^I_0(q), m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(q)$. The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I(q)$ and let α, β be scalars. Then for a given $\epsilon > 0$ we have

$$\{k \in \mathbb{N} : |x_k^/ - L_1|^{q_k} \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \ \} \in I;$$

$$\{k \in \mathbb{N} : |y_k^/ - L_2|^{q_k} \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \ \} \in I;$$

where

$$M_1 = Dmax\{1, \sup_k |\alpha|^{q_k}\};$$
$$M_2 = Dmax\{1, \sup_k |\beta|^{q_k}\};$$

and

$$D = max\{1, 2^{H-1}\} \text{ where } H = \sup_k q_k \geq 0.$$

Let

$$A_1 = \{k \in \mathbb{N} : |x_k^/ - L_1|^{q_k} < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \ \} \in \pounds(I);$$

$$A_2 = \{k \in \mathbb{N} : |y_k^/ - L_2|^{q_k} < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \ \} \in \pounds(I);$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 &= \{k \in \mathbb{N} : |(\alpha x_k^/ + \beta y_k^/) - (\alpha L_1 + \beta L_2)|^{q_k}) < \epsilon\} \\ &\supseteq \{k \in \mathbb{N} : |\alpha|^{q_k}|x_k^/ - L_1|^{q_k} < \frac{\epsilon}{2M_1}|\alpha|^{q_k} D\} \\ &\cap\{k \in \mathbb{N} : |\beta|^{q_k}|y_k^/ - L_2|^{q_k} < \frac{\epsilon}{2M_2}|\beta|^{q_k} D\}. \end{aligned}$$

Thus $A_3^c \subseteq A_1^c \cup A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(q)$. Therefore $\mathcal{Z}^I(q)$ is a linear space. The rest of the result follows similarly.

Theorem 3.2.2. Let $(q_k) \in l_\infty$. Then $m_{\mathcal{Z}}^I(q)$ and $m_{\mathcal{Z}_0}^I(q)$ are paranormed spaces, paranormed by

$$g(x) = \sup_k |x_k|^{\frac{q_k}{M}}, \text{ where } M = max\{1, \sup_k q_k\}.$$

Proof. Let $x = (x_k), y = (y_k) \in m_{\mathcal{Z}}^I(q)$.

[i] Clearly, $g(x) = 0$ if and only if $x = 0$.

[ii] $g(x) = g(-x)$ is obvious.

[iii] Since $\frac{q_k}{M} \leq 1$ and $M > 1$,using Minkowski's inequality we have

$$\sup_k |x_k + y_k|^{\frac{q_k}{M}} \leq \sup_k |x_k|^{\frac{q_k}{M}} + \sup_k |y_k|^{\frac{q_k}{M}}.$$

[iv] Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$. Let $x_k \in m_{\mathcal{Z}}^I(q)$ such that $|x_k - L|^{q_k} \geq \epsilon$. Therefore,

$$g(x - Le) = \sup_k |x_k - L|^{\frac{q_k}{M}} \leq \sup_k |x_k|^{\frac{q_k}{M}} + \sup_k |L|^{\frac{q_k}{M}},$$

where $e = (1, 1, 1.....)$. Hence

$$g(\lambda_n x_k - \lambda L) \leq g(\lambda_n x_k) + g(\lambda L) = \lambda_n g(x) + \lambda g(L),$$

as $k \to \infty$. Hence $m_{\mathcal{Z}}^I(q)$ is a paranormed space. The rest of the result follows similarly.

Theorem 3.2.3. $m_{\mathcal{Z}}^I(q)$ is a closed subspace of $l_\infty(q)$.

Proof. Let $(x_k^{(n)})$ be a Cauchy sequence in $m_{\mathcal{Z}}^I(q)$ such that $x^{(n)} \to x$. We show that $x \in m_{\mathcal{Z}}^I(q)$. Since $(x_k^{(n)}) \in m_{\mathcal{Z}}^I(q)$, then there exists a_n such that

$$\{k \in \mathbb{N} : |x^{(n)} - a_n| \geq \epsilon\} \in I.$$

We need to show that

[i] (a_n) converges to a.

[ii] If $U = \{k \in \mathbb{N} : |x_k - a| < \epsilon\}$, then $U^c \in I$.

[i] Since $(x_k^{(n)})$ is a Cauchy sequence in $m_{\mathcal{Z}}^I(q)$ then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k |x_k^{(n)} - x_k^{(i)}| < \frac{\epsilon}{3}, \quad \text{for all n, i} \geq k_0$$

For a given $\epsilon > 0$, we have

$$B_{ni} = \{k \in \mathbb{N} : |x_k^{(n)} - x_k^{(i)}| < \frac{\epsilon}{3}\},$$

$$B_i = \{k \in \mathbb{N} : |x_k^{(i)} - a_i| < \frac{\epsilon}{3}\},$$

$$B_n = \{k \in \mathbb{N} : |x_k^{(n)} - a_n| < \frac{\epsilon}{3}\}.$$

Then $B_{ni}^c, B_i^c, B_n^c \in I$.

Let

$$B^c = B_{ni}^c \cup B_i^c \cup B_n^c,$$

where

$$B = \{k \in \mathbb{N} : |a_i - a_n| < \epsilon\}.$$

Then $B^c \in I$. We choose $k_0 \in B^c$, then for each $n, i \geq k_0$, we have

$$\{k \in \mathbb{N} : |a_i - a_n| < \epsilon\} \supseteq \{k \in \mathbb{N} : |x_k^{(i)} - a_i| < \frac{\epsilon}{3}\}$$

$$\cap \{k \in \mathbb{N} : |x_k^{(n)} - x_k^{(i)}| < \frac{\epsilon}{3}\} \cap \{k \in \mathbb{N} : |x_k^{(n)} - a_n| < \frac{\epsilon}{3}\}.$$

Then (a_n) is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $a_n \to a$, as $n \to \infty$.

[ii] Let $0 < \delta < 1$ be given. Then we show that if

$$U = \{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\},$$

then $U^c \in I$. Since $x^{(n)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : |x^{(q_0)} - x| < (\frac{\delta}{3D})^M\}. \qquad [3.1]$$

which implies that $P^c \in I$.

The number q_0 can be so chosen that together with [3.1], we have

$$Q = \{k \in \mathbb{N} : |a_{q_0} - a|^{q_k} < (\frac{\delta}{3D})^M\},$$

such that $Q^c \in I$

Since

$$\{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} \geq \delta\} \in I.$$

Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} < (\frac{\delta}{3D})^M\}.$$

Let

$$U^c = P^c \cup Q^c \cup S^c,$$

where

$$U = \{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\}.$$

Therefore for each $k \in U^c$, we have

$$\{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\} \supseteq \{k \in \mathbb{N} : |x^{(q_0)} - x|^{q_k} < (\frac{\delta}{3D})^M\}$$

$$\cap \{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} < (\frac{\delta}{3D})^M\} \cap \{k \in \mathbb{N} : |a_{q_0} - a|^{q_k} < (\frac{\delta}{3D})^M\}.$$

Then the result follows.

Since the inclusions $m^I_{\mathcal{Z}}(q) \subset l_\infty(q)$ and $m^I_{\mathcal{Z}_0}(q) \subset l_\infty(q)$ are strict so in view of Theorem 2.2.3 we have the following result.

Theorem 3.2.4. The spaces $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are nowhere dense subsets of $l_\infty(q)$.

Theorem 3.2.5. The spaces $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are not separable.

Proof. We shall prove the result for the space $m^I_{\mathcal{Z}}(q)$. The proof for the other spaces will follow similarly.

Let M be an infinite subset of $\mathbb{N}$ of increasing natural numbers such that $M \in I$. Let

$$q_k = \begin{cases} 1, & \text{if } k \in M, \\ 2, & \text{otherwise.} \end{cases}$$

Let

$$P_0 = \{(x_k) : x_k = 0 \text{ or } 1, \text{for } \ k \in M \ \ and \ \ x_k = 0, \text{otherwise}\}.$$

Clearly P_0 is uncountable. Consider the class of open balls

$$B_1 = \{B(z, \frac{1}{2}) : z \in P_0\}.$$

Let C_1 be an open cover of $m^I_{\mathcal{Z}}(q)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $m^I_{\mathcal{Z}}(q)$. Thus $m^I_{\mathcal{Z}}(q)$ is not separable.

Theorem 3.2.6. Let $G = \sup_k q_k < \infty$ and I an admissible ideal. Then the following are equivalent:

[a] $(x_k) \in \mathcal{Z}^I(q)$;

[b] there exists$(y_k) \in \mathcal{Z}(q)$ such that $x_k = y_k$, for a.a.k.r.I;

[c] there exists$(y_k) \in \mathcal{Z}(q)$ and $(x_k) \in \mathcal{Z}^I_0(q)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : |y_k - L|^{q_k}) \geq \epsilon\} \in I$;

[d] there exists a subset

$$K = \{k_1 < k_2....\} \quad \text{of} \quad \mathbb{N},$$

such that $K \in \pounds(I)$ and

$$\lim_{n\to\infty} |x_{k_n} - L|^{q_{k_n}} = 0.$$

Proof.

[a] implies [b].

Let $(x_k) \in \mathcal{Z}^I(q)$. Then there exists $L \in \mathbb{C}$ such that

$$\{k \in \mathbb{N} : |x_k^{/} - L|^{q_k} \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : |x_k^{/} - L|^{q_k} \geq t^{-1}\} \in I.$$

Define a sequence (y_k) as

$$y_k = x_k, \quad \text{for all} \quad k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$,

$$y_k = \begin{cases} x_k, & \text{if } |x_k^{/} - L|^{q_k} < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in \mathcal{Z}(q)$ and from the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x_k^{/} - L|^{q_k}) \geq \epsilon\} \in I,$$

we get $x_k = y_k$, for a.a.k.r.I.

[b] implies [c].

For $(x_k) \in \mathcal{Z}^I(q)$, there exists $(y_k) \in \mathcal{Z}(q)$ such that $x_k = y_k$, for a.a.k.r.I. Let

$$K = \{k \in \mathbb{N} : x_k \neq y_k\},$$

then $k \in I$. Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_k \in \mathcal{Z}_0^I(q)$ and $y_k \in \mathcal{Z}(q)$.

[c] implies [d].

Suppose [c] holds. Let $\epsilon > 0$ be given. Let

$$P_1 = \{k \in \mathbb{N} : |z_k^{/}|^{q_k} \geq \epsilon\} \in I,$$

and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have

$$\lim_{n\to\infty} |x_{k_n}^{/} - L|^{q_{k_n}} = 0.$$

[d] implies [a].

Let

$$K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$$

and

$$\lim_{n\to\infty} |x_{k_n}^{/} - L|^{q_{k_n}} = 0.$$

Then for any $\epsilon > 0$, and Lemma 3.1.1., we have

$$\{k \in \mathbb{N} : |x_k^{/} - L|^{q_k} \geq \epsilon\} \subseteq K^c \cup \{k \in K : |x_k^{/} - L|^{q_k} \geq \epsilon\}.$$

Thus $(x_k) \in \mathcal{Z}^I(q)$.

Theorem 3.2.7. Let $h = \inf_k q_k$ and $G = \sup_k q_k$. Then the following results are equivalent.

[a] $G < \infty$ and $h > 0$.

[b] $\mathcal{Z}_0^I(q) = \mathcal{Z}_0^I$.

Proof. Suppose that $G < \infty$ and $h > 0$, then the inequalities

$$min\{1, s^h\} \leq s^{q_k} \leq max\{1, s^G\},$$

hold for any $s > 0$ and for all $k \in \mathbb{N}$. Therefore the equivalence of [a] and [b] is obvious.

Theorem 3.2.8. Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m^I_{\mathcal{Z}_0}(q) \supseteq m^I_{\mathcal{Z}_0}(r)$ if and only if $\lim_{k\in K} \inf \frac{q_k}{r_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. Let $\lim_{k\in K} \inf \frac{q_k}{r_k} > 0$ and $(x_k) \in m^I_{\mathcal{Z}_0}(r)$. Then there exists $\beta > 0$ such that $q_k > \beta r_k$, for all sufficiently large $k \in K$. Since $(x_k) \in m^I_{\mathcal{Z}_0}(r)$ for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : |x_k|^{r_k} \geq \epsilon\} \in I$$

Let $G_0 = K^c \cup B_0$ then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : |x_k|^{q_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : |x_k|^{\beta r_k}) \geq \epsilon\} \in I.$$

Therefore $(x_k) \in m^I_{\mathcal{Z}_0}(q)$. The converse part of the result follows obviously.

Theorem 3.2.9. Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m^I_{\mathcal{Z}_0}(r) \supseteq m^I_{\mathcal{Z}_0}(q)$ if and only if $\lim\limits_{k\in K}\inf \frac{r_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. The proof follows similarly as the proof of Theorem 3.2.8.

Theorem 3.2.10. Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m^I_0(r) = m^I_0(q)$ if and only if $\lim\limits_{k\in K}\inf \frac{q_k}{r_k} > 0$, and $\lim\limits_{k\in K}\inf \frac{r_k}{q_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.

Proof. By combining Theorem 3.2.8 and 3.2.9 we get the required result.